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Research Article

New Weaker Forms Of Almost Topological Spaces Under The Mapping

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. **Abstract:** In this paper, we investigate the topological properties of the image under the mapping between two topological spaces. We show then image mapping is almost topological space with certain condition subset of a given space. The space is called normal regular and normal regular compact space. Then we show that every regular compact space and for any two disjoint regularly closed subsets. is almost mildly normal compact regular space.

Keywords: Almost Compact Regular Space, Almost Completely Compact Regular Space, Normal Regular Space, Normal Compact Regular Space, Almost Normal compact Regular Space, Almost Mildly Normal Compact Regular Space.

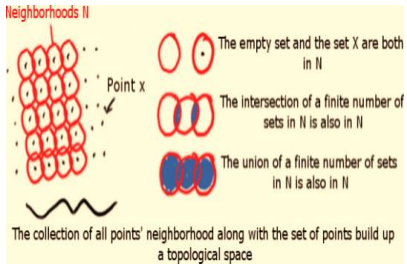
1. Introduction

Topological spaces have been investigated roughly in topological spaces, as an important and fundamental topic for the study of topology parts, several authors have presented studies of several stronger and weaker forms with continuous and multifunctional functions on certain subsets. Some use weak continuity under the name "almost continuity". And it has a great influence on topological dynamics and Banach space theory, see [1], [2], [3]. As these types of studies give different types of semi-continuous jobs. These types prompt us to further investigate the weak continuum properties of subgroups under two topological spaces.

After much research, other and new concepts have been introduced in almost weak topological spaces which are compact if each open cover of space contains closed finite subfamily. Meanwhile, the concept of continuous for weak topological approximate groups was introduced. In 1968, the concept of semi-continuous functions was introduced, and scientists identified a new class of functions called continuous functions. More research has been done on these functions. which introduced the concept of nearly weakly connected functions. The idea of a continuous weak function resulted between the topological spaces. Research and studies continued to produce the idea of semi-continuous jobs. Later on, the idea of continuous jobs was born almost weakly. Implicit is weak continuity through both approximately continuity and weak continuity independent of each other.

Later several descriptions of weak semi-continuous functions were introduced and some results improved, with some sufficient conditions for weak semi-continuous jobs to be semi-continuous. While

showing that the assumption 'semi-continuous' in many specific outcomes can be replaced by the phrase 'weakly quasi-continuous'.



2. Preliminary

2.1. Topological space

A topological space, also called an abstract topological space, is a set that meets the four conditions along with a series of open subsets:

Definition 2.1 [4]: Let $\boldsymbol{X}$ be a set. A topology on X is a set $\boldsymbol{T} \subset \boldsymbol{P}(\boldsymbol{X})$ of subsets of $\boldsymbol{X}$, called open sets, such that
(1) $\emptyset$ and $\boldsymbol{X}$ are open
(2) The intersection of finitely many open sets is open
(3) Any union of open sets is open A topological space is a set X together with a topology T on X. The members of $\boldsymbol{\tau}$ are called open sets.

Let $\boldsymbol{X}$ be a non empty set. Then $\boldsymbol{\tau_i} = \{\{\emptyset\}, \boldsymbol{X}\}$ and $\tau_d = \boldsymbol{P}(\boldsymbol{X})$ are topologies and are called as the indiscrete topology and the discrete topology, respectively. Note that if $\boldsymbol{\tau}$ is any other topology on $\boldsymbol{X}$, then $\boldsymbol{\tau_i} \subset \boldsymbol{\tau} \subset \boldsymbol{\tau_d}$.

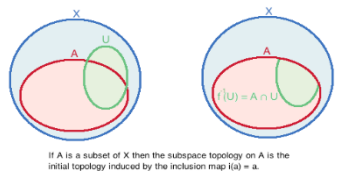
- **A topological space**

Definition 2.2 [5]: A set $\boldsymbol{S}$, together with a collection of subsets called $\boldsymbol{open\ sets}$, is called a topological space if and only if the collection of open sets satisfy the following axioms:
Axiom 1. Every open set is a set of points.
Axiom 2. The empty set is an open set.
Axiom 3. For each point $\boldsymbol{p}$, there is at least one open set containing $\boldsymbol{p}$.
Axiom 4. The union of any collection of open sets is an open set.
Axiom 5. The intersection of any finite collection of open sets is an open set.

Definition 2.3 [6]. Let $\boldsymbol{X}$ be a set. A $\boldsymbol{topology}$ on $\boldsymbol{X}$ is a collection $\boldsymbol{\tau}$ of subsets of $\boldsymbol{X}$ with the following properties.
(i) $\emptyset \in \boldsymbol{\tau}$ and $\boldsymbol{X} \in$.
(ii) Whenever $(\boldsymbol{U_i})_{i \in I}$ is a family (finite or not) of subsets of $\boldsymbol{X}$ such that $(\boldsymbol{U_i}) \in \boldsymbol{\tau}$ for all $\boldsymbol{i} \in \boldsymbol{I}$, then $\cup_{i \in I} \boldsymbol{U_i} \in \boldsymbol{\tau}$.
(iii) Whenever $\boldsymbol{U_1}; \boldsymbol{U_2} \in \boldsymbol{\tau}$, then $\boldsymbol{U_1} \cap \boldsymbol{U_2} \in \boldsymbol{\tau}$.
A $\boldsymbol{topological\ space}$ $(\boldsymbol{X}, \boldsymbol{\tau})$ is a set $\boldsymbol{X}$ together with a topology $\boldsymbol{\tau}$ on $\boldsymbol{X}$.

- **A Limit Point**

Definition 2.4 [7]: Let $\boldsymbol{A}$ be a subset of a topological space $\boldsymbol{X}$. A point $\boldsymbol{x} \in \boldsymbol{X}$ is $\boldsymbol{a\ limit\ point}$ of $\boldsymbol{A}$ if each neighborhood $\boldsymbol{U}$ of $\boldsymbol{x}$ contains a point of $\boldsymbol{A}$ other than $\boldsymbol{x}$, or equivalently, of $\boldsymbol{x}$ belongs to the closure of $\boldsymbol{A} - \{\boldsymbol{x}\}$. (**If** $\boldsymbol{x} \notin \boldsymbol{A}$**,** recall that $\boldsymbol{A} - \{\boldsymbol{x}\} = \boldsymbol{A}$) The set of limit points of $\boldsymbol{A}$ is often denoted $\boldsymbol{A'}$**,** and is called $\boldsymbol{the\ derived}$ set of **A** in $\boldsymbol{X}$.

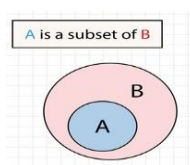


- **Subspace**

Definition 2.5 [8]: Let $\boldsymbol{X}$ be a topological space with topology $\boldsymbol{\tau}$, and let **A** be a non-empty subset of $\boldsymbol{X}$. Although it is possible to assign many different topologies to $\boldsymbol{A}$ without making any reference to the topology $\boldsymbol{\tau}$. yet we would like to assign to $\boldsymbol{A}$ a definite topology, which it inherits from $\boldsymbol{\tau}$.

A Subsets :

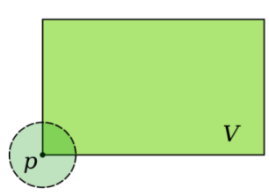
Definition 2.6 [7]. Let **X** be a set. A topology on **X** is a collection $\boldsymbol{\tau}$ of subsets of **X**, such that:
(1) $\emptyset$ and $\boldsymbol{X}$ in $\boldsymbol{\tau}$.
(2) For any $\boldsymbol{subcolliction}$ $\{\mathbf{U\alpha}\}\boldsymbol{\alpha} \in \mathbf{J}$ of $\boldsymbol{\tau}$, the union $\bigcup_{\boldsymbol{\alpha}\in\mathbf{J}} \boldsymbol{U_\alpha}$ is in $\boldsymbol{\tau}$.
(3) For any finite subcollection $\{\boldsymbol{U_1}, \dots, \boldsymbol{U_n}\}$ of $\boldsymbol{\tau}$ the intersection $\boldsymbol{U_1} \cap \dots \cap \boldsymbol{U_n}$ is in $\boldsymbol{\tau}$.
A topological space $(\mathbf{X}, \boldsymbol{\tau_X})$ is a set $\boldsymbol{X}$ with a chosen topology $\boldsymbol{\tau}$.



Definition 2.7 [9]. We say that $\boldsymbol{A}$ is **a** $\boldsymbol{subset}$ of $\boldsymbol{B}$ and we write $\boldsymbol{A} \subset \boldsymbol{B}$ or $\boldsymbol{B} \supset \boldsymbol{A}$ if every element of $\boldsymbol{A}$ is also an element of $\boldsymbol{B}$. (We also say that $\boldsymbol{A}$ is included in $\boldsymbol{B}$ or $\boldsymbol{B}$ includes $\boldsymbol{A}$ or $\boldsymbol{B}$ is a superset of $\boldsymbol{A}$.)

Definition 2.8 [10]. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are sets, then $\boldsymbol{A} - \boldsymbol{B} = \{\boldsymbol{x} \in \boldsymbol{A} : \boldsymbol{x} \notin \boldsymbol{B}\}$ is called the $\boldsymbol{complement}$ of $\boldsymbol{B}$ in $\boldsymbol{A}$. If the set $\boldsymbol{A}$ is clearly understood, we might simply refer to $\boldsymbol{A} - \boldsymbol{B}$ as the $\boldsymbol{complement}$ of $\boldsymbol{B}$. sometimes written as $\boldsymbol{B}^{\mathbf{c}}$ or $-\boldsymbol{B}$ or $\overline{\boldsymbol{B}}$.

- **Neighbourhood Points**



Definition 2.9 [11]: Let $(\boldsymbol{X}, \boldsymbol{\tau})$ be a (topological) space, $\boldsymbol{p}$ an element of $\boldsymbol{X}$ and $\boldsymbol{N}$ a subset of $\boldsymbol{X}$. We call $\boldsymbol{N}$ a $\boldsymbol{neighbourhood}$ of $\boldsymbol{p}$ (or, when we need to be fussy, a $\boldsymbol{\tau}$-$\boldsymbol{neighbourhood}$ of $\boldsymbol{p}$) if there is an open set $\boldsymbol{G}$ in $\boldsymbol{\tau}$ for which $\boldsymbol{p} \in \boldsymbol{G} \subseteq \boldsymbol{N}$.

Examples 2.1. Throughout $\boldsymbol{X}$ denotes a non-empty set.

1) $\boldsymbol{\tau} = \{\emptyset, \boldsymbol{X}\}$ is a topology on $\boldsymbol{X}$. This topology is called indiscrete topology on $\boldsymbol{X}$ and the topological space $(\boldsymbol{X}, \boldsymbol{\tau})$ is called indiscrete topological space.

2) $\boldsymbol{\tau} = \boldsymbol{P}(\boldsymbol{X})$, $\boldsymbol{P}(\boldsymbol{X})$ = power set of $\boldsymbol{X}$ is a topology on $\boldsymbol{X}$ and is called discrete topology on $\boldsymbol{X}$ and the topological space$(\boldsymbol{X}, \boldsymbol{\tau})$ is called discrete topological space.

3) Let $\boldsymbol{X} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$ then $\boldsymbol{\tau_1} = \{\emptyset, \{\boldsymbol{a}\}, \{\boldsymbol{b}, \boldsymbol{c}\}\boldsymbol{X}\}$, and $\boldsymbol{\tau_2} = \{\emptyset, \{\boldsymbol{a}\}, \{\boldsymbol{b}\}, \{\boldsymbol{a}, \boldsymbol{b}\}, \boldsymbol{X}\}$ are topologies on $\boldsymbol{X}$ whereas $\boldsymbol{\tau_3} = \{\emptyset, \{\boldsymbol{a}\}, \{\boldsymbol{b}\}, \boldsymbol{X}\}$ is a not a topology on $\boldsymbol{X}$.

4) Let $\boldsymbol{X}$ be an infinite set. Define $\boldsymbol{\tau} = \{\emptyset\} \cup \{\boldsymbol{A} \subseteq \boldsymbol{X} \mid \boldsymbol{X} - \boldsymbol{A}$ is finit$\}$ then $\boldsymbol{\tau}$ is topology on $\boldsymbol{X}$.

i) $\emptyset \in \boldsymbol{\tau}$. (by definition $\boldsymbol{\tau}$). As $\boldsymbol{X} - \boldsymbol{X} = \emptyset$, a finite set, $\boldsymbol{X} \in \boldsymbol{\tau}$.

ii) Let $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{\tau}$. If either $\boldsymbol{A} = \emptyset$ or $\boldsymbol{B} = \emptyset$, then $\boldsymbol{A} \cap \boldsymbol{B} \in \boldsymbol{\tau}$. Assume that $\boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B} \neq \emptyset$ then $\boldsymbol{X} - \boldsymbol{A}$ is finite and $\boldsymbol{X} - \boldsymbol{B}$ is finite. Hence $\boldsymbol{X} - (\boldsymbol{A} \cap \boldsymbol{B}) = (\boldsymbol{X} - \boldsymbol{A}) \cup (\boldsymbol{X} - \boldsymbol{B})$ is finite set. Therefore $\boldsymbol{A} \cap \boldsymbol{B} \in \boldsymbol{\tau}$. Thus $\boldsymbol{A}, \boldsymbol{B} \in \boldsymbol{\tau}$.

iii) Let $\boldsymbol{A_\lambda} \in \boldsymbol{\tau}$, for each $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}$ (where $\boldsymbol{\Lambda}$ is any indexing set). If each $\boldsymbol{A_\lambda} = \emptyset$, then $\bigcup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{A_\lambda} = \emptyset \in \boldsymbol{\tau}$.

If $\exists\, \boldsymbol{\lambda_0} \in \boldsymbol{\Lambda}$ such that $\boldsymbol{A_{\lambda_0}} \neq \emptyset$ then $\boldsymbol{A_{\lambda_0}} \subseteq \bigcup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{A_\lambda} \Rightarrow \boldsymbol{X} - \boldsymbol{A_{\lambda_0}} \supseteq \boldsymbol{X} - \bigcup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{A_\lambda}$.

As $\boldsymbol{X} - \boldsymbol{A_{\lambda_0}}$ is a finite set and subset of finite set being finite we get $\boldsymbol{X} - \bigcup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{A_\lambda}$ is finite and hence $\bigcup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{A_\lambda} \in \boldsymbol{\tau}$, Thus in either case, $\boldsymbol{A_\lambda} \in \boldsymbol{\tau}, \forall\, \boldsymbol{\lambda} \in \boldsymbol{\Lambda} \Rightarrow \bigcup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} \boldsymbol{A_\lambda} \in \boldsymbol{\tau}$.

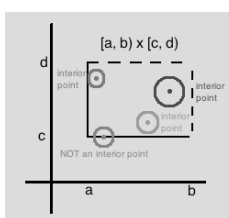
From (i), (ii) and (iii) is a topology on $\boldsymbol{X}$. This topology is called co-finite topology on $\boldsymbol{X}$ and the topological space is called co-finite topological space.

Proposition 2.1 [12] : The following properties of a topological space **X** are equivalent:

(i) The set of closed neighborhoods of any point of $\boldsymbol{X}$ is a fundamental system of neighborhoods of the point.

(ii) Given any closed subset $\boldsymbol{F}$ of $\boldsymbol{X}$ and any point $\boldsymbol{x} \notin \boldsymbol{F}$ there is a neighbourhood of $\boldsymbol{x}$ and a neighbourhood of $\boldsymbol{F}$ which do not intersect.

- **Interior Points**



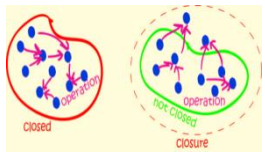
Definition 2.10 [13]: For any subset $\boldsymbol{A}$ of a topological space $\boldsymbol{X}$, the $\boldsymbol{interior}$ of $\boldsymbol{A}$ is defined to be the largest open set contained inside $\boldsymbol{A}$. It is denoted by $\boldsymbol{A^0}$.

Definition 2.11 [14] : Let $\boldsymbol{A}$ be a subset of a space $\boldsymbol{X}$. The interior of $\boldsymbol{A}$ in $\boldsymbol{X}$, denoted as $\boldsymbol{A}^\circ$ or $\boldsymbol{Int}(\boldsymbol{A})$, is the open set defined as: $\boldsymbol{Int}(\boldsymbol{A}) = \cup\, \{\boldsymbol{G} \subseteq \boldsymbol{X} : \boldsymbol{G}$ is open and $\boldsymbol{G} \subseteq \boldsymbol{A}\}$.

Theorem 2.1 [15] : Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be subsets of $\boldsymbol{X}$. Then,

(1) $\boldsymbol{Int}(\boldsymbol{A}) \subseteq \boldsymbol{A}$.

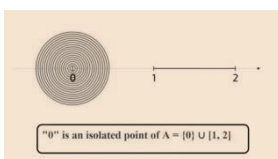
(2) If $\boldsymbol{A} \subseteq \boldsymbol{B}$, then $\boldsymbol{Int}(\boldsymbol{A}) \subseteq \boldsymbol{Int}(\boldsymbol{B})$.
(3) $\boldsymbol{Int}(\boldsymbol{X}) = \boldsymbol{X}$.
(4) $\boldsymbol{Int}(\boldsymbol{Int}(\boldsymbol{A})) = \boldsymbol{Int}(\boldsymbol{A})$.
(5) $\boldsymbol{Int}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{Int}(\boldsymbol{A}) \cap \boldsymbol{Int}(\boldsymbol{B})$.
(6) $\boldsymbol{Int}(\boldsymbol{A}) \cup \boldsymbol{Int}(\boldsymbol{B}) \subseteq \boldsymbol{Int}(\boldsymbol{A} \cup \boldsymbol{B})$.
(7) $\boldsymbol{A}$ is open iff $\boldsymbol{Int}(\boldsymbol{A}) = \boldsymbol{A}$.



- **Closure Points**

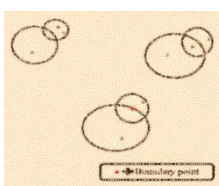
Definition2.12 [16]: Let $\boldsymbol{X}$ be a topological space and $\boldsymbol{A} \subseteq \boldsymbol{X}$. The closure of $\boldsymbol{A}$, denoted as $\overline{\boldsymbol{A}}$ or $\boldsymbol{Cl}(\boldsymbol{A})$, is the closed set defined as: $\boldsymbol{Cl}(\boldsymbol{A}) = \cap \{\boldsymbol{K} \subseteq \boldsymbol{X} : \boldsymbol{K}$ is closed and $\boldsymbol{A} \subseteq \boldsymbol{K}\}$.

Theorem 2.2 [16] : Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be subsets of $\boldsymbol{X}$. Then:
(1) $\boldsymbol{A} \subseteq \boldsymbol{Cl}(\boldsymbol{A})$.
(2) If $\boldsymbol{A} \subseteq \boldsymbol{B}$, **then** $\boldsymbol{Cl}(\boldsymbol{A}) \subseteq \boldsymbol{Cl}(\boldsymbol{B})$.
(3) $\boldsymbol{Cl}(\boldsymbol{\phi}) = \boldsymbol{\phi}$.
(4) $\boldsymbol{Cl}(\boldsymbol{Cl}(\boldsymbol{A})) = \boldsymbol{Cl}(\boldsymbol{A})$.
(5) $\boldsymbol{Cl}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{Cl}(\boldsymbol{A}) \cup \boldsymbol{Cl}(\boldsymbol{B})$.
(6) $\boldsymbol{Cl}(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \boldsymbol{Cl}(\boldsymbol{A}) \cap \boldsymbol{Cl}(\boldsymbol{B})$.
(7) $\boldsymbol{A}$ is closed iff $\boldsymbol{Cl}(\boldsymbol{A}) = \boldsymbol{A}$.



- **Isolated Points**

Definition 2.13 [13]: The closure $\overline{\boldsymbol{F}}$ of a subset $\boldsymbol{F} \subset \boldsymbol{X}$ is the smallest closed set containing . Members of $\overline{\boldsymbol{F}}$ are called $\boldsymbol{closure\ points}$ of . Closure points which are not a limit points are called $\boldsymbol{isolated\ points}$. **(**It is clear that $\boldsymbol{F'} \subset \overline{\boldsymbol{F}}$**).**



- **The Boundary Points**

Definition 2.14 [13]: $\boldsymbol{The\ boundary}$ of $\boldsymbol{A} \subset \boldsymbol{X}$ is defined by $\boldsymbol{\partial A} = \overline{\boldsymbol{A}} \cap \overline{(\boldsymbol{Ac})}$.

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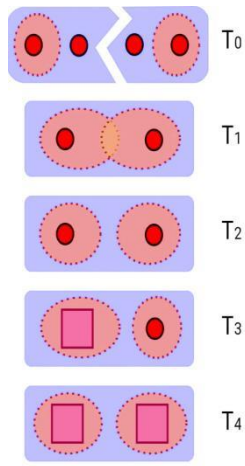
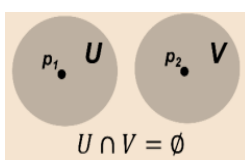


- **Separation Axioms**

Definition 2.15 [17]: Let $(\boldsymbol{X}, \boldsymbol{T})$ be a topological space. The following are called $\boldsymbol{Separation\ Axioms}$.
(i) $\boldsymbol{X}$ is called $\boldsymbol{T_o}$ if for every pair of distinct points $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ there exists an open set $\boldsymbol{U}$ containing $\boldsymbol{x}$ which does not contain $\boldsymbol{y}$.
(ii) $\boldsymbol{X}$ is called $\boldsymbol{T_1}$ if one point sets are closed.
(iii) $\boldsymbol{X}$ is called $\boldsymbol{T_2}$ or Hausdorff if $\boldsymbol{X}$ is $\boldsymbol{T_1}$ and for any pair of distinct points $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ there exist disjoint open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ where $\boldsymbol{x} \in \boldsymbol{U}$ and $\boldsymbol{y} \in \boldsymbol{V}$.
(iv) $\boldsymbol{X}$ is called $\boldsymbol{T_3}$ or regular if $\boldsymbol{X}$ is $\mathbf{T_1}$ and for any point $\boldsymbol{x} \in \boldsymbol{X}$ and any closed set $\boldsymbol{C}$ not containing $\boldsymbol{x}$ there exist open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ where $\boldsymbol{x} \in \boldsymbol{U}$ and $\boldsymbol{C} \subseteq \boldsymbol{V}$.
(v) $\boldsymbol{X}$ is called $\boldsymbol{completely\ regular}$ if for any point $\boldsymbol{x} \in \boldsymbol{X}$ and closed set $\boldsymbol{C}$ not containing $\boldsymbol{x}$ there exists a continuous function $\boldsymbol{f} : \boldsymbol{X} \to [\mathbf{0}, \mathbf{1}]$ where $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$ and $\boldsymbol{f}(\boldsymbol{C}) = \mathbf{1}$.
(vi) $\boldsymbol{X}$ is called $\boldsymbol{T_4}$ or normal if $\boldsymbol{X}$ is $\mathbf{T_1}$ and for any pair of disjoint closed sets $\boldsymbol{C}$ and $\boldsymbol{D}$ there exist disjoint open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ where $\boldsymbol{C} \subseteq \boldsymbol{U}$ and $\boldsymbol{D} \subseteq \boldsymbol{V}$



2.2. Compact space

Definition 2.16 [19] : A space $\boldsymbol{Y}$ is $\boldsymbol{Hausdorff}$ (or separated) if each two distinct points have nonintersecting nbds, that is whenever $\boldsymbol{p} \neq \boldsymbol{q}$ there are nbds $\boldsymbol{U}(\boldsymbol{p}), \boldsymbol{V}(\boldsymbol{q})$ such that $\boldsymbol{U} \cap \boldsymbol{V} = \emptyset$.
Proof [50]. ($\mathbf{T_2} \Rightarrow \mathbf{T_1}$) ($\mathbf{T_1} \Rightarrow \mathbf{T_0}$). Let $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{y} \in \{\boldsymbol{x}\}^{\boldsymbol{c}}$. Since $\boldsymbol{X}$ is $\mathbf{T_2}$ there exist disjoint open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ where $\boldsymbol{x} \in \boldsymbol{U}$ and $\boldsymbol{y} \in \boldsymbol{V}$.

Theorem 2.3 [20] : A topological space is a Hausdorff space if and only if for each point $\boldsymbol{a}$ the intersection of all closed neighborhoods of a is the set $\{\boldsymbol{a}\}$.
Proof. Let $\boldsymbol{X}$ be a Hausdorff space and let $\boldsymbol{b} \neq \boldsymbol{a}$. Choose disjoint open sets $\boldsymbol{\mathcal{O}_\alpha}$, and $\boldsymbol{\mathcal{O}_b}$, such that $\boldsymbol{a} \in \boldsymbol{\mathcal{O}_\alpha}$, and $\boldsymbol{b} \in \boldsymbol{\mathcal{O}_b}$. Then $\boldsymbol{a} \in \boldsymbol{\mathcal{O}_\alpha} \subseteq {}_{\mathbf{c}}\boldsymbol{\mathcal{O}_b}$ but $\boldsymbol{b} \notin {}_{\mathbf{c}}\boldsymbol{\mathcal{O}_b}$ and so ${}_{\mathbf{c}}\boldsymbol{\mathcal{O}_b}$ is a closed neighborhood of $\boldsymbol{a}$ not containing $\boldsymbol{b}$. Hence the intersection of all closed neighborhoods of $\boldsymbol{a}$ does not contain any $\boldsymbol{b} \neq \boldsymbol{a}$. Conversely, if this intersection is $\{\boldsymbol{a}\}$, then given any $\boldsymbol{b} \neq \boldsymbol{a}$ there exist an open $\boldsymbol{\mathcal{O}_\alpha}$, and a closed $\boldsymbol{C_a}$,

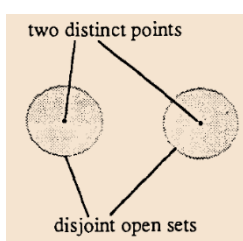
such that $\boldsymbol{a} \in \mathcal{O}_{\alpha} \subseteq \boldsymbol{C_a}$ and $\boldsymbol{b} \notin \boldsymbol{C_a}$. Then $\mathcal{O}_{\boldsymbol{b}} = {}_{c}\mathcal{O}_{\boldsymbol{a}}$ is open, $\boldsymbol{b} \in \mathcal{O}_{\boldsymbol{b}}$, and $\mathcal{O}_{\alpha} \cap \mathcal{O}_{\boldsymbol{b}} = \emptyset$. Hence Hausdorff's axiom holds in $\boldsymbol{X}$. There exist $(\boldsymbol{T_1})$ spaces which are not $(\mathbf{T_2})$ spaces. For instance, let $\boldsymbol{X}$ be an infinite set and let the topology on $\boldsymbol{X}$ be the topology of finite complements. For any pair of distinct points $\boldsymbol{a}, \boldsymbol{b}$ the open sets $\mathcal{O}_{\boldsymbol{b}} = \boldsymbol{X} - \{\boldsymbol{a}\}$ and $\mathcal{O}_{\alpha} = \boldsymbol{X} - \{\boldsymbol{b}\}$ fulfill the requirements of axiom $(\boldsymbol{T_1})$ but not of $(\boldsymbol{T_2})$. Since any two nonvoid open sets have a nonvoid intersection, there exists no pair $\mathcal{O}_{\alpha}$, $\mathcal{O}_{\boldsymbol{b}}$ which would satisfy the requirements of axiom $(\boldsymbol{T_2})$.

Theorem 2.3 [18] : Every finite point set in a Hausdorff space $\boldsymbol{X}$ is closed.
Proof: It suffices to show that every one-point set $\{\boldsymbol{x_0}\}$ is closed. If $\boldsymbol{x}$ is a point of $\boldsymbol{X}$ different from $\boldsymbol{x_0}$, then $\boldsymbol{x}$ and $\boldsymbol{x_0}$ have disjoint neighborhoods $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. Since $\boldsymbol{U}$ does not intersect $\{\boldsymbol{x_0}\}$,the point $\boldsymbol{x}$ cannot belong to the closure of the set $\{\boldsymbol{x_0}\}$**.** As a result, the closure of the set $\{\boldsymbol{x_0}\}$ is $\{\boldsymbol{x_0}\}$ itself, so that it is closed.

Theorem 2.4 [21] : If $\boldsymbol{X}$ is a Hausdorff space and $\boldsymbol{A} \subset \boldsymbol{X}$, then $\boldsymbol{A}$ with the subspace topology is a Hausdorff space. If $\{\boldsymbol{X_i}: \boldsymbol{i} \in \boldsymbol{I}\}$ is a $\boldsymbol{family\ of\ Hausdorff\ spaces}$, then $\prod_{i \in I} \boldsymbol{X_i}$ is Hausdorff.
Proof. Suppose that $\boldsymbol{a}, \boldsymbol{b}$ are distinct points in $\boldsymbol{A}$. Because $\boldsymbol{X}$ is Hausdorff, there are disjoint open sets $\boldsymbol{U}, \boldsymbol{V}$ in $\boldsymbol{X}$ with $\boldsymbol{a} \in \boldsymbol{U}, \boldsymbol{b} \in \boldsymbol{V}$. Then $\boldsymbol{U} \cap \boldsymbol{A}, \boldsymbol{V} \cap \boldsymbol{A}$ are disjoint open sets in A with the subspace topology and $\boldsymbol{a} \in \boldsymbol{U} \cap \boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{V} \cap \boldsymbol{A}$, showing that $\boldsymbol{A}$ is Hausdorff. Suppose that $\boldsymbol{x}, \boldsymbol{y}$ are distinct elements of $\prod_{i \in I} \boldsymbol{X_i}$. $\boldsymbol{x}$ and $\boldsymbol{y}$ being distinct means there is some $\boldsymbol{i} \in \boldsymbol{I}$ such that $\boldsymbol{x}(\boldsymbol{i}) \neq \boldsymbol{y}(\boldsymbol{i})$. Then $\boldsymbol{x}(\boldsymbol{i}), \boldsymbol{y}(\boldsymbol{i})$ are distinct points in $\boldsymbol{X_i}$, which is Hausdorff, so there are disjoint open sets $\boldsymbol{U_i}, \boldsymbol{V_i}$ in $\boldsymbol{X_i}$ with $\boldsymbol{x}(\boldsymbol{i}) \in \boldsymbol{U_i}, \boldsymbol{y}(\boldsymbol{i}) \in \boldsymbol{V_i}$. Let U = $\boldsymbol{\pi_i}^{-1}(\boldsymbol{U_i})$, **V =** $\boldsymbol{\pi_i}^{-1}(\boldsymbol{V_i})$, where $\boldsymbol{\pi_i}$ is the projection map from the product to $\boldsymbol{X_i}$. $\boldsymbol{U}$ and $\boldsymbol{V}$ are disjoint, and $\boldsymbol{x} \in \boldsymbol{U}, \boldsymbol{y} \in \boldsymbol{V}$, showing that $\prod_{i \in I} \boldsymbol{X_i}$ is Hausdorff.

Theorem 2.5 [15] : If a topological space satisfies axioms $\boldsymbol{(T_0)}$ and $\boldsymbol{(T_3)}$**,** then it is a Hausdorff space.
Proof. Given any pair of distinct points $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{X}$, by axiom $(\boldsymbol{T_0})$ there exists an open set $\mathcal{O}$ which separates these points. We may assume that $\boldsymbol{a} \notin \mathcal{O}$ and $\boldsymbol{b} \in \mathcal{O}$. We apply axiom $(\boldsymbol{T_3})$ with $\boldsymbol{A} = \boldsymbol{c}\mathcal{O}$ and find open sets $\mathcal{O}_{\boldsymbol{A}}$, and $\mathcal{O}_{\boldsymbol{b}}$, such that $\boldsymbol{A} \subseteq \mathcal{O}_{\boldsymbol{A}}$, $\boldsymbol{b} \in \mathcal{O}_{\boldsymbol{b}}$, and $\mathcal{O}_{\boldsymbol{A}} \cap \mathcal{O}_{\boldsymbol{b}} = \emptyset$. Therefore $\in \boldsymbol{A} \subseteq \mathcal{O}_{\boldsymbol{A}}$, and $\boldsymbol{b} \in \mathcal{O}_{\boldsymbol{b}}$, This shows that $\boldsymbol{X}$ is a Hausdorff space.

Examples 2.2
1. Any discrete topological space $(\boldsymbol{X}, \boldsymbol{\tau})$ with $|\boldsymbol{X}| \geq \mathbf{2}$ is a $\boldsymbol{T_2} - \boldsymbol{space}$.
Foe $\boldsymbol{x} \neq \boldsymbol{y}$ in $\boldsymbol{X}$, $\{\boldsymbol{x}\}$ and $\{\boldsymbol{y}\}$ are two disjoint open sets containing $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.
2. $(\mathbb{R}, \boldsymbol{\tau_u})$ is a $\boldsymbol{T_2} - \boldsymbol{space}$.
Let $\boldsymbol{x} \neq \boldsymbol{y}$ in $\mathbb{R}$, Then $|\boldsymbol{x} - \boldsymbol{y}| = \boldsymbol{r} > 0$. $(\boldsymbol{x} - \frac{\boldsymbol{r}}{\mathbf{3}}, \boldsymbol{x} + \frac{\boldsymbol{r}}{\mathbf{3}})$ and $(\boldsymbol{y} - \frac{\boldsymbol{r}}{\mathbf{3}}, \boldsymbol{y} + \frac{\boldsymbol{r}}{\mathbf{3}})$ are disjoint open sets in $\mathbb{R}$ containing $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively. Hence $(\mathbb{R}, \boldsymbol{\tau_u})$ is a $\boldsymbol{T_2} - \boldsymbol{space}$.



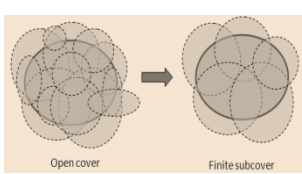
Preposition 2.2 [12] : Let $\boldsymbol{X}$ be a topological space. Then the following statements are equivalent:

(i) Any two distinct points of $\boldsymbol{X}$ have disjoint neighbourhood. (Hausdorff) .
(ii) The intersection of the closed neighbourhood of any point of $\boldsymbol{X}$ consists of that point alone.
(iii) The diagonal of the product space $\boldsymbol{X} \times \boldsymbol{X}$ is a closed set.
(iv) For every set $\boldsymbol{I}$, the diagonal of the product space $\boldsymbol{Y} = \boldsymbol{X}^{\boldsymbol{I}}$ is closed in $\boldsymbol{Y}$.

Theorem 2.6 [7] : Let $\boldsymbol{A}$ be a subset of a Hausdorff space $\boldsymbol{X}$. A point $\boldsymbol{x} \in \boldsymbol{X}$ is a limit point of $\boldsymbol{A}$ if and only if each neighborhood $\boldsymbol{U}$ of $\boldsymbol{x}$ meets $\boldsymbol{A}$ in infinitely many points.
Proof. If $\boldsymbol{U} \cap \boldsymbol{A}$ consists of infinitely many points, then it certainly contains other points than $\boldsymbol{x}$, so $\boldsymbol{U}$ meets $\boldsymbol{A} - \{\boldsymbol{x}\}$.
Conversely, if $\boldsymbol{U} \cap \boldsymbol{A}$ is finite, then then $\boldsymbol{U} \cap (\boldsymbol{A} - \{\boldsymbol{x}\}) = \{\boldsymbol{x}_1, \dots . \boldsymbol{x}_n\}$ is closed, so $\boldsymbol{V} = \boldsymbol{U} - \{\boldsymbol{x}_1, \dots . \boldsymbol{x}_n\} = \boldsymbol{U} \cap (\boldsymbol{X} - \{\boldsymbol{x}_1, \dots . \boldsymbol{x}_n\})$ is open. Then $\boldsymbol{x} \in \boldsymbol{V}$,$\boldsymbol{V}$ is open, and $\boldsymbol{V} \cap (\boldsymbol{A} - \{\boldsymbol{x}\}) = \emptyset$,so $\boldsymbol{x}$ is not a limit point of $\boldsymbol{A}$.

Theorem 2.7 [7] : Each finite subset $\boldsymbol{A} \subset \boldsymbol{X}$ in a Hausdorff space is closed.
Proof. The set $\boldsymbol{A}$ is the union of a finite collection of singleton sets $\{\boldsymbol{x}\}$, so it suffices to prove that each singleton set $\{\boldsymbol{x}\}$ is closed in $\boldsymbol{X}$.
Consider any other point $\boldsymbol{y} \boldsymbol{\epsilon} \boldsymbol{X}$, with $\boldsymbol{x} \neq \boldsymbol{y}$. By the Hausdorff property there are open subsets $\boldsymbol{U}, \boldsymbol{V} \subset \boldsymbol{X}$ with $\boldsymbol{x} \in \boldsymbol{U}$ and $\in \boldsymbol{V}$, such that $\boldsymbol{U} \cap \boldsymbol{V} = \emptyset$. Then $\notin \boldsymbol{V}$, so $\boldsymbol{X} - \boldsymbol{V}$ is a closed set that contains $\{\boldsymbol{x}\}$. Hence $\{\boldsymbol{x}\} \subset \boldsymbol{X} - \boldsymbol{V}$, so $\boldsymbol{y} \notin \boldsymbol{Cl}\{\boldsymbol{x}\}$. Since $\boldsymbol{Cl}\{\boldsymbol{x}\}$ cannot contain any other points than $\boldsymbol{x}$, it follows that $\{\boldsymbol{x}\} = \boldsymbol{Cl}\{\boldsymbol{x}\}$ and$\{\boldsymbol{x}\}$ is closed.

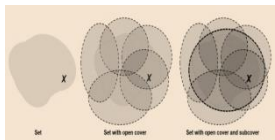
Claim 2.1 [22]. Every subspace of a Hausdorff space is Hausdorff.
Proof. Suppose $\boldsymbol{X}$ is a Hausdorff space, and $\boldsymbol{A}$ is a subspace of $\boldsymbol{X}$. The spoiler now picks two distinct points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{A}$, and the prover must demonstrate that there are disjoint sets containing $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively that are open in the subspace topology for $\boldsymbol{A}$. Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are distinct points of $\boldsymbol{X}$, the prover has at his disposal disjoint open sets of $\boldsymbol{X}$, namely $\boldsymbol{U}$ and $\boldsymbol{V}$, such that $\boldsymbol{U}$ contains $\boldsymbol{x}$ and $\boldsymbol{V}$ contains $\boldsymbol{y}$. He them picks $\boldsymbol{U} \cap \boldsymbol{A}$ and $\boldsymbol{V} \cap \boldsymbol{A}$ as subsets of $\boldsymbol{A}$. Clearly:
• $\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{x} \in \boldsymbol{U}$, so $\boldsymbol{x} \in \boldsymbol{U} \cap \boldsymbol{A}$. Similarly, $\boldsymbol{y} \in \boldsymbol{V} \cap \boldsymbol{A}$.
• $\boldsymbol{U}$ and $\boldsymbol{V}$ themselves are disjoint, so $\boldsymbol{U} \cap \boldsymbol{A}$ and $\boldsymbol{V} \cap \boldsymbol{A}$ are disjoint.
• $\boldsymbol{U} \cap \boldsymbol{A}$ is open relative to $\boldsymbol{A}$, because it is the intersection with $\boldsymbol{A}$ of an open set in $\boldsymbol{X}$. Similarly: $\boldsymbol{V} \cap \boldsymbol{A}$ is also open relative to $\boldsymbol{A}$. Hence, the prover has managed to pick disjoint open sets containing $\boldsymbol{x}$ and $\boldsymbol{y}$, relative to the subspace topology of $\boldsymbol{A}$.



Definition 2.17 [23] : A $\boldsymbol{covering}$ of a topological space $\boldsymbol{X}$ is a collection of sets whose union is $\boldsymbol{X}$. It is an $\boldsymbol{open\ covering}$ if the sets are open. A $\boldsymbol{subcover}$ is a subset of this collection which still covers the space.
If $\boldsymbol{A} \subset \boldsymbol{X}$ then, for convenience, we sometimes use "$\boldsymbol{cover\ A}$" for a collection of subsets of $\boldsymbol{X}$ whose union contains $\boldsymbol{A}$.

Examples 2.3. Let $X = \mathbb{R}$ and let $Un = (-n, +n), for\ n = 1, 2, \ldots$. Then $\{Un \mid n \in \mathbf{N}\}$ is an open cover of $\mathbb{R}$, i.e. $\mathbb{R} = \bigcup_{n=1}^{\infty} U_n$. This is so because, for $r \in \mathbb{R}$ we can choose a positive integer m greater than $|r|$ and then $r \in (-m, +m) \subset U_m$, so $r \in U_m \subset \bigcup_{n=1}^{\infty} U_n$.
Hence $\bigcup_{n=1}^{\infty} U_n$ contains every real number, i.e. $\bigcup_{n=1}^{\infty} U_n = R$.
A subcover of this open cover is $\{U_n \mid n \in J\}$ where J is the set of even positive integers.

Example 2.4. Let $X = [0, 1]$ (with the subspace topology induced from $\mathbb{R}$). Let $U_1 = [0, 1/4)$ and $U_n = (1/n, 1]$, for $n = 2, 3, 4, \ldots$. Then $U_1 = [0, 1] \cap (-1/4, 1/4)$ is open in the subspace topology and so is $U_n = [0, 1] \cap (1/n, 2)$, for $n \geq 2$. Note that $\{U_n \mid n = 1, 2, \ldots\}$ is an open cover and $\{U_1, U_5\}$ is a subcover.



Definition 2.18 [23] : A topological space X is said to be $compact$ if every open covering of X has a finite subcover. (This is sometimes referred to as the Heine-Borel property.)

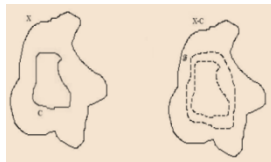
Examples 2.5.
1. Any subset of an indiscrete topological space (X, τ) is compact, as $\{X\}$ is the only open cover for any $E \subseteq X$.
2. Any finite subset of any topological space (X, τ) is compact.
Co-finite topological space is compact.

Theorem 2.8 [10] : The following are equivalent for any topological space (X, τ)
(1) X is compact.
(2) Every family $\mathcal{F}$ of closed sets in X with the finite intersection property also has $\bigcap \mathcal{F} = \emptyset$.
Proof $1) \Rightarrow 2)$ Suppose X is compact and that $\mathcal{F}$ is a family of closed sets with **FIP** (the finite intersection property) Let $u = \{X - F : F \in \mathcal{F}\}$. For any $F_1, F_2, \ldots, F_n \in \mathcal{F}$. the **FIP** tells us that $X \neq X - \bigcap_{i=1}^{n} F_n = (X - F_1) \cup \ldots \cup (X - F_n)$. In other words, no finite subcollection of $u\ covers\ X$. Since X is compact, u cannot be a cover of cover X, that is, $X = \cup u = \cap \{X - U : U \in u\} = \cap \mathcal{F} \neq \emptyset$.

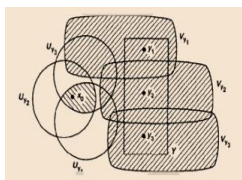
Lemma 2.1 [19] : Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.
Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}\alpha \in J$ is a covering of Y by sets open in X. Then the collection. $\{A_\alpha \cap Y \mid \alpha \in J$ is a covering of Y by sets open in Y; hence a finite subcollection $(A_{\alpha 1} \cap Y, \ldots, A_{\alpha n} \cap Y)$ covers Y. Then $(A_{,,}, \ldots, A_{,,}]$ is a subcollection of A that covers Y. Conversely, suppose the given condition holds; we wish to prove Y compact. Let $A' = \{A'_\alpha\}$ be a covering of Y by sets open in Y. For each α choose a set , open in X such that $A'_\alpha = A, \cap Y$. The collection $\mathcal{A} = (A,)$ is a covering of Y by sets open in X. By hypothesis, some finite subcollection $\{A_{,,}, \ldots, A_{,,}]$ covers Y . Then $\{A'_{\alpha 1}, \ldots, A'_{\alpha n}\}$ is a subcollection of A' that covers.

Theorem 2.9 [19] : Every closed subspace of a compact space is compact.

Proof. Let $\boldsymbol{Y}$ be a closed subspace of the compact space $\boldsymbol{X}$. Given a covering $\boldsymbol{A}$ of $\boldsymbol{Y}$ by sets open in $\boldsymbol{X}$, let us form an open covering $\boldsymbol{\mathcal{B}}$ of $\boldsymbol{X}$ by adjoining to $\boldsymbol{\mathcal{A}}$ the single open set $\boldsymbol{X} - \boldsymbol{Y}$, that is, $\boldsymbol{\mathcal{B}} = \boldsymbol{\mathcal{A}}\ \boldsymbol{U}\ (\boldsymbol{X} - \boldsymbol{Y})$ .Some finite subcollection of covers $\boldsymbol{X}$. If this subcollection contains the set $\boldsymbol{X} - \boldsymbol{Y}$, discard $\boldsymbol{X} - \boldsymbol{Y}$; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of. $\boldsymbol{\mathcal{A}}$ that covers $\boldsymbol{Y}$.



Theorem 2.10 [24] : A closed subset of a compact set is also compact.
Proof. Take a compact space $\boldsymbol{X}$ with a closed subset C let F be a family of open subsets of $\boldsymbol{X}$ such that $\boldsymbol{C} \subset \cup\, \mathfrak{F}$. Add the open set $\boldsymbol{U} = \boldsymbol{X} - \boldsymbol{C}$ to F and we get a cover of $\boldsymbol{X}$. $\boldsymbol{X}$ is compact so $\mathfrak{F} \cup \boldsymbol{U}$ admits a finite subcover $\boldsymbol{O}_1 \cup \boldsymbol{O}_2 \cup \ldots \cup \boldsymbol{O}_k \cup (\boldsymbol{X} - \boldsymbol{C})$. $C \subset \boldsymbol{O}_1 \cup \boldsymbol{O}_2 \cup \ldots \cup \boldsymbol{O}_k$ since $(\boldsymbol{X} - \boldsymbol{C}) \cap \boldsymbol{C} = \emptyset$ and this new set $\{\ \boldsymbol{O}_1 \cup \boldsymbol{O}_2 \cup \ldots \cup \boldsymbol{O}_k$ is a finite subcover of $\boldsymbol{C}$.



Theorem 2.11 [19] : Every compact subspace of a Hausdorff space is closed.
Proof. Let $\boldsymbol{Y}$ be a compact subspace of the Hausdorff space $\boldsymbol{X}$. We shall prove that $\boldsymbol{X} - \boldsymbol{Y}$ is open, so that $\boldsymbol{Y}$ is closed. Let $\boldsymbol{x_o}$ be a point of $\boldsymbol{X} - \boldsymbol{Y}$. We show there is a neighborhood of $\boldsymbol{x_o}$ that is disjoint from $\boldsymbol{Y}$. For each point $\boldsymbol{y}$ of $\boldsymbol{Y}$, let us choose disjoint neighborhoods $\boldsymbol{U}_{\mathrm{y}}$ and $\boldsymbol{V}_{\mathrm{y}}$ of the points $\boldsymbol{x_o}$ and $\boldsymbol{y}$, respectively (using the Hausdorff condition). The collection $\{\boldsymbol{V}_{\mathrm{y}} \mid \boldsymbol{y} \in \boldsymbol{Y}\}$ is a covering of $\boldsymbol{Y}$ by sets open in $\boldsymbol{X}$; therefore, finitely many of them $\boldsymbol{V}_{y1}, \ldots, \boldsymbol{V}_{yn}$ cover $\boldsymbol{Y}$. The open set $\boldsymbol{V} = \boldsymbol{V}_{y1} \cap \ldots \cap \boldsymbol{V}_{yn}$ contains $\boldsymbol{Y}$, and it is disjoint from the open set $\boldsymbol{U} = \boldsymbol{U}_{y1} \cap \ldots \cap \boldsymbol{U}_{yn}$ formed by taking the intersection of the corresponding neighborhoods of $\boldsymbol{x_o}$. For if $\boldsymbol{z}$ is a point of $\boldsymbol{V}$,then $\boldsymbol{z} \in \boldsymbol{V}_{yi}$ for some i, hence $\boldsymbol{z} \notin \boldsymbol{U}_{yi}$ and so $\boldsymbol{z} \notin \boldsymbol{U}$. See Figure. Then $\boldsymbol{U}$ is a neighborhood of $\boldsymbol{x_o}$ disjoint from $\boldsymbol{Y}$, as desired.

Lemma 2.2 [19] : If $\boldsymbol{Y}$ is a compact subspace of the Hausdorff space $\boldsymbol{X}$ and $\boldsymbol{x_o}$ is not in $\boldsymbol{Y}$, then there exist disjoint open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ of $\boldsymbol{X}$ containing $\boldsymbol{x_o}$ and $\boldsymbol{Y}$, respectively.

Theorem 2.12 [23] : The unit interval $\boldsymbol{I} = [\boldsymbol{0}, \boldsymbol{1}]$ is compact.
Proof. Let $\boldsymbol{U}$ be an open covering of $\boldsymbol{I}$. Put $\boldsymbol{S} = \{\boldsymbol{s} \in \boldsymbol{I} \mid [\boldsymbol{0}, \boldsymbol{s}]$ is covered by a finite subcollection of $\boldsymbol{U}\}$. Let $\boldsymbol{b}$ the least upper bound of $\boldsymbol{S}$. Clearly S must be an interval of the form $\boldsymbol{S} = [\boldsymbol{0}, \boldsymbol{b})$ or $\boldsymbol{S} = [\boldsymbol{0}, \boldsymbol{b}]$. In the former case, however, consider a set $\boldsymbol{U} \in \mathbf{U}$ containing the point $\boldsymbol{b}$. This set must contain an interval of the form $[\boldsymbol{a}, \boldsymbol{b}]$. But then we can throw $\boldsymbol{U}$ in with the hypothesized finite cover of $[\boldsymbol{0}, \boldsymbol{a}]$ to obtain a finite cover of $[\boldsymbol{0}, \boldsymbol{b}]$. Thus we must have that $\boldsymbol{S} = [\boldsymbol{0}, \boldsymbol{b}]$ for some $\boldsymbol{b} \in [\boldsymbol{0}, \boldsymbol{1}]$. But if $\boldsymbol{b} < \boldsymbol{1}$, then a similar argument shows that there is a finite cover of $[\boldsymbol{0}, \boldsymbol{c}]$ for some $\boldsymbol{c} > \boldsymbol{b}$, contradicting the choice of $\boldsymbol{b}$. Thus $\boldsymbol{b} = \boldsymbol{1}$ and we have found the desired finite cover of $[\boldsymbol{0}, \boldsymbol{1}]$.

2.3 . Topological mapping

- **A Functions**

Definition 2.18 [8]. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two non-empty sets (which may be equal). A rule $\boldsymbol{f}$ which assigns to each element of $\boldsymbol{X}$ a unique element of $\boldsymbol{Y}$ is called a function or (single-valued) mapping (or ma) from $\boldsymbol{X}$ to $\boldsymbol{Y}$. If $\boldsymbol{x} \in \boldsymbol{X}$ then the element $\boldsymbol{y}$ of $\boldsymbol{Y}$ assigned to $\boldsymbol{x}$, under the rule $\boldsymbol{f}$, is called the image of $\boldsymbol{x}$, or the value of f at the point x. If $\boldsymbol{y}$ is the image of $\boldsymbol{x}$, we write $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$.

Definition 2.19 [9]. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be sets. A relation $\boldsymbol{f} \subset \boldsymbol{X} \times \boldsymbol{Y}$ is called *a* $\boldsymbol{function}$ from $\boldsymbol{X}$ to $\boldsymbol{Y}$ if

(i) $\boldsymbol{D}(\boldsymbol{f}) = \boldsymbol{X}$,

(ii)$\forall \boldsymbol{x} \in \boldsymbol{X}$ .The set { $\boldsymbol{y} \in \boldsymbol{Y} : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{f}$ } has exactly one element.

- **A Continuous Functions**

Definition 2.20 [8]: A function $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is called $\boldsymbol{one-one}$ or $\boldsymbol{injective}$ if distinct elements of $\boldsymbol{X}$ have distinct image in $\boldsymbol{Y}$ (i.e.) whenever $\boldsymbol{f}(\boldsymbol{x_1}) = \boldsymbol{f}(\boldsymbol{x_2})$ $\boldsymbol{for\ x_1, x_2} \in \boldsymbol{X}$ then $\boldsymbol{x_1} = \boldsymbol{x_2}$.

Definition 2.21 [8]: A function $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is called $\boldsymbol{onto}$ or $\boldsymbol{surjective}$ if elements of **Y** is the image of some element of $\boldsymbol{X}$, (i.e.) if the range of $\boldsymbol{f}$ is $\boldsymbol{Y}$.

Definition 2.22 [8]: A function $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is called $\boldsymbol{bijective}$ if it is both $\boldsymbol{one-one}$ ($\boldsymbol{injective}$) and $\boldsymbol{onto}$ ($\boldsymbol{surjective}$).

Definition 2.23 [8]. Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ be a function from a topological space $\boldsymbol{X}$ to a topological space $\boldsymbol{Y}$. Let $\boldsymbol{x_0} \in \boldsymbol{X}$. Then $\boldsymbol{f}$ is said to be continuous at $\boldsymbol{x_0}$ If and only if for every open set $\boldsymbol{V}$ containing $\boldsymbol{f}(\boldsymbol{x_0})$ in $\boldsymbol{Y}$. there exist an open set $\boldsymbol{u}$ in $\boldsymbol{X}$ such that $\boldsymbol{x} \in \boldsymbol{u} \subseteq \boldsymbol{f^{-1}}(\boldsymbol{V})$.

Examples 2.6.

1. Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{X}^*$ be a function. Let $\boldsymbol{\alpha} \in \boldsymbol{X}^*$ be any fixed point.
Define $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{\alpha}$, for each in $\boldsymbol{X}$. $\boldsymbol{f}$ is continuous at each $\boldsymbol{x} \in \boldsymbol{X}$ since for every open set $\boldsymbol{G}^*$ containing $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{\alpha}$, here is an open set $\boldsymbol{G} = \boldsymbol{X}$ containing $\boldsymbol{x}$ such that $\boldsymbol{f}(\boldsymbol{G}) \subseteq \boldsymbol{G}^*$. Hence $\boldsymbol{f}$ is continues on $\boldsymbol{X}$.
2. Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{X}^*$ be a function. Let $\boldsymbol{\alpha} \in \boldsymbol{X}$ such that $\{\boldsymbol{a}\} \in \boldsymbol{\tau}$. Then is $\boldsymbol{f}$ continuous at $\boldsymbol{\alpha}$.
Let $\boldsymbol{G}^* \in \boldsymbol{\tau}^*$ such that $\boldsymbol{f}(\boldsymbol{a}) \in \boldsymbol{G}^*$. Then $\boldsymbol{\alpha} \in \boldsymbol{f^{-1}}(\boldsymbol{G}^*) \Rightarrow \{\boldsymbol{a}\} \subseteq \boldsymbol{f^{-1}}(\boldsymbol{G}^*)$. Define $\boldsymbol{G} = \{\boldsymbol{a}\}$. Then $\boldsymbol{G} \in \boldsymbol{\tau}$ such that $a \in G$ and $\boldsymbol{f}(\boldsymbol{G}) \subseteq \boldsymbol{G}^*$. Hence $\boldsymbol{f}$ is continues on $\boldsymbol{a}$.

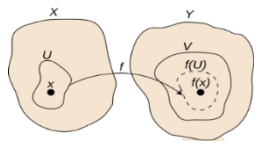
Definition 2.24 [25]: Suppose $\boldsymbol{f} : \boldsymbol{A} \to \boldsymbol{B}$ and $\boldsymbol{g} : \boldsymbol{B} \to \boldsymbol{C}$ are functions with the property that the codomain of $\boldsymbol{f}$ equals the domain of $\boldsymbol{g}$. $\boldsymbol{The\ composition}$ of $\boldsymbol{f}$ with $\boldsymbol{g}$ is another function, denoted as $\boldsymbol{g}\circ\boldsymbol{f}$ and defined as follows: If $\boldsymbol{x} \in \boldsymbol{A}$, then $\boldsymbol{g}\circ\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))$. Therefore $\boldsymbol{g}\circ\boldsymbol{f}$ sets elements of $\boldsymbol{A}$ to elements of $\boldsymbol{C}$, so $\boldsymbol{g}\circ\boldsymbol{f}: \boldsymbol{A} \to \boldsymbol{C}$.

Definition 2.25 [25]: For a set $\boldsymbol{A}$, ***the identity function*** on $\boldsymbol{A}$ is the function $\boldsymbol{i_A : A \to A}$ defined as $\boldsymbol{i_A(x) = x}$ for every $\boldsymbol{x \in A}$.

Definition 2.26 [25]: Given a relation $\boldsymbol{R}$ from $\boldsymbol{A}$ to $\boldsymbol{B}$, ***the inverse relation*** of $\boldsymbol{R}$ is the relation from $\boldsymbol{B}$ to A defined as $\boldsymbol{R^{-1} = \{(y,x): (x,y) \in R\}}$. In other words, the inverse of $\boldsymbol{R}$ is the relation $\boldsymbol{R^{-1}}$ obtained by interchanging the elements in every ordered pair in $\boldsymbol{R}$.

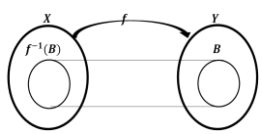
Definition 2.27 [25]: If $\boldsymbol{f : A \to B}$ is bijective then its ***inverse*** is the function $\boldsymbol{f^{-1} : B \to A}$. The functions $\boldsymbol{f}$ and $\boldsymbol{f^{-1}}$ obey the equations $\boldsymbol{f^{-1} \circ f = i_A}$ and $\boldsymbol{f \circ f^{-1} = i_B}$.

Definition 2.29 [8]. Let $\boldsymbol{f: X \to Y}$ be a function from a topological space $\boldsymbol{X}$ to a topological space $\boldsymbol{Y}$. Let $\boldsymbol{x_0 \in X}$. Then $\boldsymbol{f}$ is said to be ***continuous*** at $\boldsymbol{x_0}$ If and only if for every open set $\boldsymbol{V}$ containing $\boldsymbol{f(x_0)}$ in $\boldsymbol{Y}$. there exist an open set $\boldsymbol{u}$ in $\boldsymbol{X}$ such that $\boldsymbol{x \in u \subseteq f^{-1}(V)}$.



Proposition 2.3 [23]. A function $\boldsymbol{f: X \to Y}$ between topological spaces is continuous ⇔ it is continuous at each point $\boldsymbol{x \in X}$.

Definition 2.30 [8]. the function $\boldsymbol{f^{-1}: Y \to X}$ is called the inverse of $\boldsymbol{f: X \to Y}$

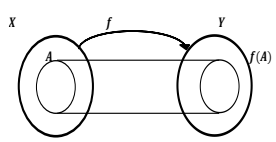


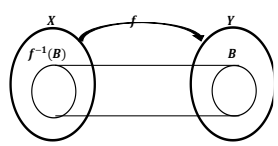
Theorem 2.14 [14]. Suppose $\boldsymbol{f: (X,\tau) \to (Y,\tau')}$. The following are equivalent.

1. is continuous.
2. if $\boldsymbol{\mathcal{O} \in \tau'}$ then $\boldsymbol{f^{-1}[\mathcal{O}] \in \tau}$ (the inverse image of an open set is open).
3. if $\boldsymbol{F}$ is closed in $\boldsymbol{Y}$, then $\boldsymbol{f^{-1}[F]}$ is closed in $\boldsymbol{X}$. (the inverse image of a closed set is closed)
4. for every $\boldsymbol{A \subset X : f[cl_X(A)] \subset cl_Y(f[A])}$).

Definition 2.31 [8]: Let $\boldsymbol{f : X \to Y}$ be a function.

(i) If $\boldsymbol{A}$ is a subset of $\boldsymbol{X}$, then its image set $\boldsymbol{f(A)}$ is a subset of $\boldsymbol{Y}$ defined by $\boldsymbol{f(A) = \{f(x): x \in A\}}$.





(ii) If $\boldsymbol{B}$ is a subset of $\boldsymbol{Y}$ **,** then is inverse image $\boldsymbol{f^{-1}(B)}$ is the subset of $\boldsymbol{X}$ defined by $\boldsymbol{f^{-1}(B) = \{x : f(x) \in B\}}$

Examples 2.7. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be any two non-empty sets and let $\boldsymbol{f: X \to Y}$ be any function. Let $\boldsymbol{\tau}$ be topology on $\boldsymbol{Y}$. Define $\boldsymbol{\tau^* = \{f^{-1}(G) | \, G \in \tau\}}$, Where $\boldsymbol{f^{-1}(G) = \{x \in X | \, f(x) \in G\}}$ Then $\boldsymbol{\tau^*}$ is topology on **X**.

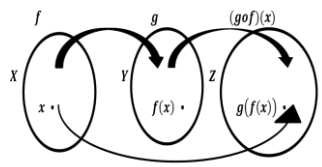
1. $\boldsymbol{f^{-1}(\emptyset) = \emptyset \Rightarrow \emptyset \in \tau^*}$ and $\boldsymbol{f^{-1}(Y) = X \Rightarrow X \in \tau^*}$**.**

2. Let $\boldsymbol{f^{-1}(G) \in \tau^*}$ and $\boldsymbol{f^{-1}(H) \in \tau^*}$, For $\boldsymbol{H \in \tau}$. Then $\boldsymbol{f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)}$ and $\boldsymbol{G, H \in \tau}$ will imply $\boldsymbol{f^{-1}(G) \cap f^{-1}(H) \in \tau^*}$**.**

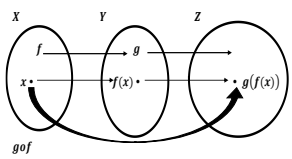
3. Let $\boldsymbol{f^{-1}(G_\lambda) \in \tau^* \; \forall \, \lambda \in \Lambda}$**,** where $\boldsymbol{\Lambda}$ any indexing set is. Then

$\boldsymbol{f^{-1}\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda)}$. As $\boldsymbol{\bigcup_{\lambda \in \Lambda} G_\lambda \in \tau}$, we get $\boldsymbol{\bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda) \in \tau^*}$.

Thus from (i), (ii) and (iii) we get $\boldsymbol{\tau^*}$ is a topology on $\boldsymbol{X}$.



Definition 2.32 [8]: Let $\boldsymbol{f: X \to Y}$ and $\boldsymbol{g: Y \to Z}$ be any two functions. The composition (or resultant) $\boldsymbol{(gof)}$ of this functions is defined to be the function from $\boldsymbol{X}$ to $\boldsymbol{Z}$ given by $\boldsymbol{(gof)(x) = g(f(x))}$.



Theorem 2.15 [8]: The composition $\boldsymbol{(gof) : X \to Z}$ to tow objective functions $\boldsymbol{f : X \to Y}$ and $\boldsymbol{g : Y \to Z}$ is objective function.

Proof. We shall first prove that $\boldsymbol{gof}$ is one to one**.** Let $\boldsymbol{x_1, x_2 \in X}$ then $\boldsymbol{(gof)(x_1) = (gof)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow f(x_1) = f(x_2)}$ for $\boldsymbol{g}$ is one to one. $\Rightarrow \boldsymbol{x_1 = x_2}$ as $\boldsymbol{f}$ is one to one. Hence $\boldsymbol{(gof)}$ is one to one.

Theorem 2.16 [8]: Let $\boldsymbol{f : X \to Y}$ and $\boldsymbol{g : Y \to Z}$ be tow continuous functions. Then $\boldsymbol{(gof) : X \to Z}$ is a continuous function.

Proof. Let $\boldsymbol{h = gof}$ to prove that $\boldsymbol{h}$ is continuous we have to establish that for every open set $\boldsymbol{W}$ in , $\boldsymbol{h^{-1}(W)}$ is open in $\boldsymbol{X}$**.**

Examples 2.8.

1. Let $\boldsymbol{X = \{a, b, c\}}$, $\boldsymbol{\tau = \{\emptyset, \{a\}, X\}}$, $\boldsymbol{X^* = \{p, q, r\}}$ and $\boldsymbol{\tau^* = \{\emptyset, \{p\}, \{p, r\}, X^*\}}$.

i) Define $\boldsymbol{f: X \to X^*}$, by $\boldsymbol{f(a) = p}$, $\boldsymbol{f(b) = q}$ and $\boldsymbol{f(c) = r}$. Then $\boldsymbol{f}$ is an open map (Note that $\boldsymbol{f}$ is not a continuous map).

ii) Define $\boldsymbol{g: X \to X^*}$, by $\boldsymbol{g(a) = p}$, $\boldsymbol{g(b) = q}$ and $\boldsymbol{g(c) = r}$. Then $\boldsymbol{g}$ is an open map (Note that $\boldsymbol{g}$

is not a continuous map).

2. Let (X, τ) be any topological space. Let $X^* = \{a, b, c\}$ and $\tau^* = \{\emptyset, \{a\}, \{a, c\}, X^*\}$.

i) Define $\boldsymbol{\theta}: \boldsymbol{X} \to \boldsymbol{X}^*$, by $\boldsymbol{\theta}(\boldsymbol{x}) = \boldsymbol{a}$, $\forall\, x \in X$. Then $\boldsymbol{\theta}$ is an open map but not a closed map.

ii) Define $\boldsymbol{\varphi}: \boldsymbol{X} \to \boldsymbol{X}^*$, by $\boldsymbol{\varphi}(\boldsymbol{x})\ \boldsymbol{a}$, $\forall\, x \in X$. Then $\boldsymbol{\varphi}$ is closed map but not an open map.

Preposition 2.4 [12]. Let $\boldsymbol{f}, \boldsymbol{g}$ be two continuous mappings of a topological space $\boldsymbol{X}$ into a Hausdorff space $\boldsymbol{Y}$. then the set of all $\boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$ is closed in $\boldsymbol{X}$.

Corollary 2.1 [12]: Let $\boldsymbol{f}, \boldsymbol{g}$ be two continuous mappings of a topological space $\boldsymbol{X}$ into a Hausdorff space $\boldsymbol{Y}$. If $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x})$ at all points of a dense subset of $\boldsymbol{X}$, then $\boldsymbol{f} = \boldsymbol{g}$.

Corollary 2.2 [12]: If $\boldsymbol{f}$ is a continuous mapping of a topological space $\boldsymbol{X}$ into a Hausdorff space $\boldsymbol{Y}$, then the graph of $\boldsymbol{f}$ is closed in $\boldsymbol{X} \times \boldsymbol{Y}$. For this graph is the set of all $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{X} \times \boldsymbol{Y}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$, and the two mappings $(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{y}$ and $(\boldsymbol{x}, \boldsymbol{y}) \to \boldsymbol{f}(\boldsymbol{x})$ are continuous.

Theorem 2.17 [8]:

(a) The continuous image of a compact space is compact.

(b) A compact subset of Hausdorff space is closed.

(c) A closed subset of a compact space is compact.

(d) A compact Hausdorff space is normal.

Theorem 2.18 [14] : For any space $(\boldsymbol{X}, \boldsymbol{\tau})$

(a) If $\boldsymbol{X}$ is compact and $\boldsymbol{F}$ is closed in $\boldsymbol{X}$, then $\boldsymbol{F}$ is compact.

(b) If $\boldsymbol{X}$ is a Hausdorff space and $\boldsymbol{K}$ is a compact subset, then $\boldsymbol{K}$ is closed in $\boldsymbol{X}$.

Proof. $\boldsymbol{a}$) Suppose $\boldsymbol{F}$ is a closed set in a compact space , and let $\boldsymbol{u} = \{\cup\, \boldsymbol{U}_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \boldsymbol{U}\}$ be a cover of $\boldsymbol{F}$ by sets open in $\boldsymbol{F}$. For each $\boldsymbol{\alpha}$, pick an open set $\boldsymbol{V}_{\boldsymbol{\alpha}}$ in $\boldsymbol{X}$ such that $\boldsymbol{U}_{\boldsymbol{\alpha}} = \boldsymbol{V}_{\boldsymbol{\alpha}} \cap \boldsymbol{F}$. Since $\boldsymbol{F}$ is closed, the collection $\boldsymbol{v} = \{\boldsymbol{X} - \boldsymbol{F}\} \cup \{\boldsymbol{V}_{\boldsymbol{\alpha}}: \boldsymbol{\alpha} \in \boldsymbol{A}\}$ is an open cover of . Therefore we can find $\boldsymbol{V}_{\boldsymbol{\alpha}1}, \dots, \boldsymbol{V}_{\boldsymbol{\alpha}n}$ so that $\{\boldsymbol{X} - \boldsymbol{F}, \boldsymbol{V}_{\boldsymbol{\alpha}1}, \dots, \boldsymbol{V}_{\boldsymbol{\alpha}n}\}$ covers $\boldsymbol{X}$. Then $\{\boldsymbol{U}_{\boldsymbol{\alpha}1}, \dots, \boldsymbol{U}_{\boldsymbol{\alpha}2}\}$ covers $\boldsymbol{F}$, so $\boldsymbol{F}$ is compact.

$\boldsymbol{b}$) Suppose $\boldsymbol{K}$ is a **compact** set in a Hausdorff space$\boldsymbol{X}$. Let $\boldsymbol{y} \in \boldsymbol{X} - \boldsymbol{K}$ For each $\boldsymbol{x} \in \boldsymbol{K}$, we can choose disjoint open sets $\boldsymbol{U}_{\boldsymbol{x}}$ and $\boldsymbol{V}_{\boldsymbol{x}}$ in $\boldsymbol{X}$ where $\boldsymbol{x} \in \boldsymbol{U}_{\boldsymbol{x}}$ and $\boldsymbol{y} \in \boldsymbol{V}_{\boldsymbol{x}}$. Then $\boldsymbol{u} = \{\boldsymbol{U}_{\boldsymbol{x}} : \boldsymbol{x} \in \boldsymbol{K}\}$ covers $\boldsymbol{K}$ so there are finitely many points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \boldsymbol{K}$ such that. Then , $\boldsymbol{U}_{\boldsymbol{x}1}, \dots, \boldsymbol{U}_{\boldsymbol{x}n}$ covers $\boldsymbol{K}$, then $\boldsymbol{y} \in \boldsymbol{V} = \boldsymbol{V}_{\boldsymbol{x}1} \cap \dots \cap \boldsymbol{V}_{\boldsymbol{x}n} \subseteq \boldsymbol{X} - \boldsymbol{K}$, so $\boldsymbol{X} - \boldsymbol{K}$ is open and $\boldsymbol{K}$ is closed.

Theorem 2.19 [23] : If $\boldsymbol{X}$ is compact and $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is continuous, then $\boldsymbol{f}(\boldsymbol{X})$ is compact.

Proof. We may as well replace $\boldsymbol{Y}$ by $\boldsymbol{f}(\boldsymbol{X})$ and so assume that $\boldsymbol{f}$ is onto. For any open cover of $\boldsymbol{Y}$ look at the inverse images of its sets and apply the compactness of $\boldsymbol{X}$.

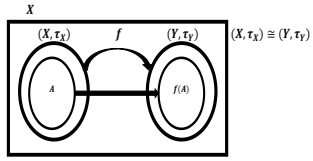
Theorem 2.20 [24] : The continuous image of a compact space is compact.

Proof: Suppose we have $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ an onto continuous function with $\boldsymbol{X}$ compact.

We must show $\boldsymbol{Y}$ is also compact. Take any open cover, $\mathfrak{F}$ of $\boldsymbol{Y}$. Since $\boldsymbol{f}$ is continuous for each open set $\boldsymbol{O} \in \mathfrak{F}$ $\boldsymbol{f}^{-1}(\boldsymbol{O})$ is open in $\boldsymbol{X}$. Define the set $\boldsymbol{\vartheta} = \{\boldsymbol{f}^{-1}(\boldsymbol{O}) | \forall\, \boldsymbol{O} \in \mathfrak{F}\}$.

Which is an open cover of $\boldsymbol{X}$. $\boldsymbol{X}$ is compact, therefore $\boldsymbol{\vartheta}$ admits a finite subcover.

$\boldsymbol{X} = \cup_{\boldsymbol{i}=1}^{\boldsymbol{k}} \boldsymbol{f}^{-1}(\boldsymbol{O}_{\mathrm{i}})$.



- **Homeomorphism**

Definition 2.33 [23]: A function $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ between topological spaces is called a $\boldsymbol{homeomorphism}$ if $\boldsymbol{f^{-1}} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ exists (i.e., $\boldsymbol{f}$ is one-one and onto) and both $\boldsymbol{f}$ and $\boldsymbol{f^{-1}}$ are continuous. The notation $\boldsymbol{X} \approx \boldsymbol{Y}$ means that $\boldsymbol{X}$ is homeomorphic to $\boldsymbol{Y}$.

Theorem 2.21 [8]: Let $\boldsymbol{X} \simeq \boldsymbol{Y}$ mean that there exists a homeomorphism from $\boldsymbol{X}$ onto $\boldsymbol{Y}$. Then:

1. $\boldsymbol{X} \simeq \boldsymbol{Y}\ \forall\ \boldsymbol{X} \in \boldsymbol{\tau}$

2. $\boldsymbol{X} \simeq \boldsymbol{Y} \Rightarrow \boldsymbol{Y} \simeq \boldsymbol{X}, \forall\ \boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{\tau}$

3. $\boldsymbol{X} \simeq \boldsymbol{Y}$ and $\boldsymbol{Y} \simeq \boldsymbol{Z} \Rightarrow \boldsymbol{X} \simeq \boldsymbol{Z}, \forall\ \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \in \boldsymbol{\tau}$

Proof.

(1) let $\boldsymbol{i_X} : \boldsymbol{X} \rightarrow \boldsymbol{X}$ be the identity function on a topological space X. Then $\boldsymbol{i_X}$ is homeomorphism. By definition $\boldsymbol{i_X}(\boldsymbol{X}) = \boldsymbol{X}, \forall\ \boldsymbol{x} \in \boldsymbol{X}$ **,** it follows that $\boldsymbol{i_X}$ is bijective. Let $\boldsymbol{u}$ be an open set in $\boldsymbol{X}$. Then $\boldsymbol{i_X}^{-1}(\boldsymbol{u}) = \boldsymbol{u}$ and $\boldsymbol{i_x}(\boldsymbol{u}) = \boldsymbol{u}$**.** Therefore, $\boldsymbol{i_X}$ is bicontinuous. Hence $\boldsymbol{i_X}$ a homeomorphism. Hence $\boldsymbol{X} \simeq \boldsymbol{Y}$.

(2) Let $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ be a homeomorphism**.** Then $\boldsymbol{f^{-1}} : \boldsymbol{Y} \rightarrow \boldsymbol{X}$ exists and is continuous. Since $(\boldsymbol{f^{-1}})^{-1} = \boldsymbol{f}$ is continuous and $\boldsymbol{f^{-1}}$ is bijective, it follows that $\boldsymbol{f^{-1}}$ is a homeomorphism from $\boldsymbol{Y}$ onto $\boldsymbol{X}$**.** Hence $\boldsymbol{X} \simeq \boldsymbol{Y} \Rightarrow \boldsymbol{Y} \simeq \boldsymbol{X}$**.**

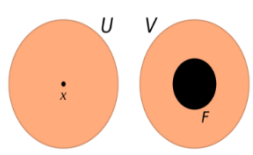
(3) Let $\boldsymbol{f} : \boldsymbol{X} \rightarrow \boldsymbol{Y}$ and $\boldsymbol{f} : \boldsymbol{Y} \rightarrow \boldsymbol{Z}$ be a homeomorphism. Then $\boldsymbol{gof} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ is also continuous (By definition homeomorphism). Since $\boldsymbol{f^{-1}}$ and $\boldsymbol{g^{-1}}$ are continuous, $\boldsymbol{f^{-1}og^{-1}} = (\boldsymbol{gof})^{-1}$ is also continuous. The function $\boldsymbol{gof} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ is bijective because the resultant of tow bijective function is also a bijective function. Hence $\boldsymbol{gof} : \boldsymbol{X} \rightarrow \boldsymbol{Z}$ is a homeomorphism**.** Hence $\boldsymbol{X} \simeq \boldsymbol{Y}$ and $\boldsymbol{Y} \simeq \boldsymbol{Z} \Rightarrow \boldsymbol{X} \simeq \boldsymbol{Z}$.

Theorem 2.22 [7] : A one-to-one, onto, and continuous function from a compact space $\boldsymbol{X}$ to a Hausdorff space $\boldsymbol{Y}$ is a homeomorphism.

Proof: Let $\boldsymbol{f}: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ be a function as described above, and take a closed subset of $\boldsymbol{X}$, $\boldsymbol{C}.\boldsymbol{C}$ is compact by (by definition compact) and $\boldsymbol{f}(\boldsymbol{C})$ must be compact (by definition compact), it is also closed in $\boldsymbol{Y}$. Our function f takes closed sets to closed sets, proving $\boldsymbol{f^{-1}}$ is continuous, making $\boldsymbol{f}$ a homeomorphism.

- **Completely Hausdorff Spaces**

- **Definition 2.34 [26]:** A topological space $\boldsymbol{X}$ is called **Completely Hausdorff Spaces** or if whenever $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$, $\boldsymbol{x} \neq \boldsymbol{y}$, there is a continuous function $\boldsymbol{f} : \boldsymbol{X} \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ and $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{1}$. Every completely Hausdorff space is Hausdorff.



2.4. Regular space and Normal space

Definition 2.35 [27]: A topological space $\boldsymbol{X}$ is called $\boldsymbol{regular}$ if whenever $\boldsymbol{F}$ is a closed set and $\boldsymbol{x} \notin \boldsymbol{F}$, containing $\boldsymbol{x}$, there exist disjoint open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ such that $\boldsymbol{x} \in \boldsymbol{U}$ and $\boldsymbol{F} \subseteq \boldsymbol{V}$.

Definition 2.36 [11] : We call $(\boldsymbol{X}, \boldsymbol{\tau})$ $\boldsymbol{T_3}$ or $\boldsymbol{regular}$ if
(i) it is $\boldsymbol{T_1}$ and
(ii)for each closed subset $\boldsymbol{F}$ of $\boldsymbol{X}$ and each point $\boldsymbol{x} \in \boldsymbol{X} \setminus \boldsymbol{F}, \exists\, \boldsymbol{G}, \boldsymbol{H} \in \boldsymbol{\tau}$ such that $\boldsymbol{x} \in \boldsymbol{G}, \boldsymbol{F} \subseteq \boldsymbol{H}, \boldsymbol{G} \cap \boldsymbol{H} = \emptyset$

Definition 2.37 [16] : Let $\boldsymbol{A}$ be a subset of a topological space $\boldsymbol{X}$. Then $\boldsymbol{A}$ is called a $\boldsymbol{regular\ open}$ set if $\boldsymbol{A} = \boldsymbol{Int}(\boldsymbol{Cl}(\boldsymbol{A}))$. A set $\boldsymbol{A}$ is called $\boldsymbol{regular\ closed}$ if $\boldsymbol{A^c}$ is regular open; that is, $\boldsymbol{A} = \boldsymbol{Cl}(\boldsymbol{Int}(\boldsymbol{A}))$.

Examples 2.9.
1. Every discrete $\boldsymbol{T-space}$ $(\boldsymbol{X}, \boldsymbol{\tau})$ with $|\boldsymbol{X}| \geq \boldsymbol{1}$ is a regular space.
2. Let **X= {a, b, c}**, $\boldsymbol{\tau} = \{\emptyset, \{\boldsymbol{a}\}, \{\boldsymbol{b}, \boldsymbol{c}\}, \boldsymbol{X}\}$ The topological space$(\boldsymbol{X}, \boldsymbol{\tau})$ is a regular space.
The family of closed sets in $(\boldsymbol{X}, \boldsymbol{\tau})$ is $\boldsymbol{k} = \{\emptyset, \boldsymbol{X}, \{\boldsymbol{a}\}, \{\boldsymbol{b}, \boldsymbol{c}\}\}$**.**
<u>**Case 1**</u>: $\boldsymbol{a} \notin \{\boldsymbol{b}, \boldsymbol{c}\}$**.** Then take $\boldsymbol{G} = \{\boldsymbol{a}\}$ and $\boldsymbol{H} = \{\boldsymbol{b}, \boldsymbol{c}\}$. $\boldsymbol{G}, \boldsymbol{H} \in \boldsymbol{\tau}$**.** $\boldsymbol{G} \cap \boldsymbol{H} = \emptyset$**,** $\{\boldsymbol{a}\} \in \boldsymbol{G}$ and $\{\boldsymbol{b}, \boldsymbol{c}\} \in \boldsymbol{H}$**.**
<u>**Case 2**</u>**:** $\boldsymbol{b} \notin \{\boldsymbol{a}\}$**.** Take $\boldsymbol{G} = \{\boldsymbol{b}, \boldsymbol{c}\}$ and $\boldsymbol{H} = \{\boldsymbol{a}\}$**.** Then $\boldsymbol{G}, \boldsymbol{H} \in \boldsymbol{\tau}$**.** $\boldsymbol{G} \cap \boldsymbol{H} = \emptyset$**,** $\boldsymbol{b} \in \{\boldsymbol{b}, \boldsymbol{c}\} \subseteq \boldsymbol{G}$ and $\{\boldsymbol{a}\} \subseteq \boldsymbol{F}$.
<u>**Case 3**</u>**:** $\boldsymbol{c} \notin \{\boldsymbol{a}\}$**.** Take $\boldsymbol{G} = \{\boldsymbol{b}, \boldsymbol{c}\}$ and $\boldsymbol{H} = \{\boldsymbol{a}\}$**.** Then $\boldsymbol{G}, \boldsymbol{H} \in \boldsymbol{\tau}$**.** $\boldsymbol{G}, \boldsymbol{c} \in \boldsymbol{G}, \{\boldsymbol{a}\} \subseteq \boldsymbol{H}$ and $\boldsymbol{G} \cap \boldsymbol{H} = \emptyset$**.**
Thus given a closed set $\boldsymbol{F}$ and a point $\boldsymbol{p} \notin \boldsymbol{F}$there exist disjoint open sets one containing $\boldsymbol{p}$ and the other containing $\boldsymbol{F}$. This shows that the $\boldsymbol{T-space}$ $(\boldsymbol{X}, \boldsymbol{\tau})$ is a regular space.
3. $(\mathbb{R}, \boldsymbol{\tau_u})$ is a regular space: Let $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|$ and $\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{r}) = (\boldsymbol{x} - \boldsymbol{r}, \boldsymbol{x} + \boldsymbol{r})$. Then the topology $\boldsymbol{\tau_u}$ is induced by the metric $\boldsymbol{d}$ on $\boldsymbol{R}$. $(\mathbb{R}, \boldsymbol{\tau_u})$ is a regular space.

Theorem 2.23 [16] : If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both regular open sets, then $\boldsymbol{A} \cap \boldsymbol{B}$ is regular open set.

Theorem 2.24 [16] : Let $\boldsymbol{X}$ be a topological space and $\boldsymbol{A}, \boldsymbol{B}$ subsets of $\boldsymbol{X}$. Then the following statements are equivalent:
(1) $\boldsymbol{A}$ is regular open set.
(2) $\boldsymbol{A} = \boldsymbol{Int}(\boldsymbol{Cl}(\boldsymbol{U}))$ for some open set $\boldsymbol{U}$.
(3) $\boldsymbol{A} = \boldsymbol{Ext}(\boldsymbol{O})$ for some open set $\boldsymbol{O}$.
(4) $\boldsymbol{A} = \boldsymbol{Int}(\boldsymbol{C})$ for some closed set $\boldsymbol{C}$.
And all the following statements are equivalent:
(1) $\boldsymbol{A}$ is regular closed set.
(2) $\boldsymbol{A} = \boldsymbol{Cl}(\boldsymbol{Int}(\boldsymbol{C}))$ for some closed set $\boldsymbol{C}$.
(3) $\boldsymbol{A} = \boldsymbol{Cl}(\boldsymbol{U})$ for some regular open set $\boldsymbol{U}$.
(4) $\boldsymbol{A} = \boldsymbol{Cl}(\boldsymbol{O})$ for some open set $\boldsymbol{O}$.

Lemma 2.3 [11] : A $\mathbf{T_1}$ space $(\boldsymbol{X}, \boldsymbol{\tau})$ is $\mathbf{T_3} \Leftrightarrow$ given $\boldsymbol{x} \in$ open $\boldsymbol{G}$, $\exists$ open $\boldsymbol{H}$ such that $\boldsymbol{x} \in \boldsymbol{H} \subseteq \overline{\boldsymbol{H}} \subseteq \boldsymbol{G}$.

Theorem 2.25 [27]. The following are equivalent for any space $\boldsymbol{X}$:

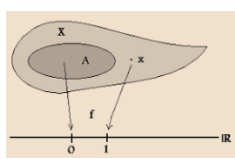
(i) $\boldsymbol{X}$ is regular .
(ii) if $\boldsymbol{O}$ is an open set containing $\boldsymbol{x}$, then there exists an open set $\boldsymbol{U} \subseteq \boldsymbol{X}$ such that $\boldsymbol{x} \in \boldsymbol{U} \subseteq \boldsymbol{cl\, U} \subseteq \boldsymbol{O}$.
(iii) at each point $\boldsymbol{x} \in \boldsymbol{X}$ there exists a neighborhood base consisting of closed neighborhoods**.**
Proof. $\boldsymbol{i}) \Longrightarrow \boldsymbol{ii})$ Suppose $\boldsymbol{X}$ is regular and $\boldsymbol{O}$ is an open set with $\boldsymbol{x} \in \boldsymbol{O}$. Letting $\boldsymbol{F} = \boldsymbol{X} - \boldsymbol{O}$, we use regularity to get disjoint open sets $\boldsymbol{U}, \boldsymbol{V}$ with $\boldsymbol{x} \in \boldsymbol{U}$ and $\boldsymbol{F} \subseteq \boldsymbol{V}$ as illustrated below**.** Then $\boldsymbol{x} \in \boldsymbol{U} \subseteq \boldsymbol{cl\, U} \subseteq \boldsymbol{O}$ (since $U \subseteq X - V$).
$\boldsymbol{ii}) \Rightarrow \boldsymbol{iii})$ If $\boldsymbol{N} \in \boldsymbol{N}_{\mathbf{x}}$, then $\boldsymbol{x} \in \boldsymbol{O} = \boldsymbol{int\, N}$. By $\boldsymbol{ii})$, we can find an open set $\boldsymbol{U}$ so that $\boldsymbol{x} \in \boldsymbol{U} \subseteq \boldsymbol{cl\, U} \subseteq \boldsymbol{O} \subseteq N$ Since $\boldsymbol{cl\, U}$ is a neighborhood of $\boldsymbol{x}$, the closed neighborhoods of $\boldsymbol{x}$ form a neighborhood base at $\boldsymbol{x}$.
$\boldsymbol{iii}) \Rightarrow \boldsymbol{i})$ Suppose $\boldsymbol{F}$ is closed and $\notin \boldsymbol{F}$. (By $\boldsymbol{ii})$, there is a closed neighborhood $\boldsymbol{K}$ of $\boldsymbol{x}$ such $\boldsymbol{x} \in \boldsymbol{K} \subseteq \boldsymbol{X} - \boldsymbol{F}$. We can choose $\boldsymbol{U} = \boldsymbol{int\, K}$ and $\boldsymbol{V} = \boldsymbol{X} - \boldsymbol{K}$ to complete the proof that $\boldsymbol{X}$ is regular.

Proposition 2.5 [12] : if every point $\boldsymbol{x}$ of a topological space $\boldsymbol{X}$ has a closed neighbourhood which is a regular subspace of $\boldsymbol{X}$, then $\boldsymbol{X}$ is regular.

Proposition 2.6 [12] : Every subspace of a regular space is regular.

Corollary 3.1 [12] : Every compact space is regular.

- **Completely Regular Space**



Definition 2.38 [20] : We call $(\boldsymbol{X}, \boldsymbol{\tau})$ $\boldsymbol{T}_{3\frac{1}{2}}$ or **Completely Regular Space** or **Tychonoff** if:
(i) it is $\boldsymbol{T}_{\mathbf{1}}$ and
(ii)for each closed (non-empty) subset $\boldsymbol{F}$ of $\boldsymbol{X}$ and each point $\boldsymbol{x} \in \boldsymbol{X} \backslash \boldsymbol{F}$, $\exists\, \boldsymbol{continuous}\, \boldsymbol{f} : \boldsymbol{X} \rightarrow [\mathbf{0}, \mathbf{1}]$ such that $\boldsymbol{f}\,(\boldsymbol{x}) = \mathbf{1}$ and $\boldsymbol{f}\,(\boldsymbol{F}) = \{\mathbf{0}\}$.

Examples 2.10.
1. Every metric space is a completely regular space.
2. $(\boldsymbol{X}, \boldsymbol{\tau_u})$ is a completely regular space.

Theorem 2.26 [27] : $\mathbf{T}_{3\frac{1}{2}} \Longrightarrow \mathbf{T}_{\mathbf{3}}$ $(\Longrightarrow \mathbf{T}_{\mathbf{2}} \Longrightarrow \mathbf{T}_{\mathbf{1}} \Longrightarrow \mathbf{T}_{\mathbf{0}})$
Proof. Suppose is a closed set in not containing $\boldsymbol{x}$. If $\boldsymbol{X}$ is $\mathbf{T}_{3\frac{1}{2}}$, we can choose $\boldsymbol{f} \in \boldsymbol{C}(\boldsymbol{X})$ with $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$ and $\boldsymbol{f} \backslash \boldsymbol{F} = \mathbf{1}$. Then $\boldsymbol{U} = \boldsymbol{f}^{-\mathbf{1}}\left[\left(-\infty, \frac{\mathbf{1}}{\mathbf{2}}\right)\right]$ and $\boldsymbol{f}^{-\mathbf{1}}\left[\left(\frac{\mathbf{1}}{\mathbf{2}}, \infty\right)\right]$ are disjoint open sets with $\boldsymbol{x} \in \boldsymbol{U}, \boldsymbol{F} \subseteq \boldsymbol{V}$. Therefore $\boldsymbol{X}$ is regular. Since $\boldsymbol{X}$ is $\boldsymbol{T}_{\mathbf{1}}$, $\boldsymbol{X}$ is $\boldsymbol{T}_{\mathbf{3}}$.

- **Normal Space**

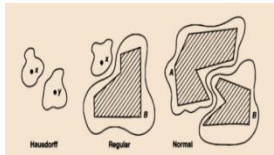
Definition 2.39 [11] : We call (X,τ) $\mathbf{T_4}$ or $\boldsymbol{normal}$ if :
(i) it is T_1 and
(ii)for each two disjoint closed (non-empty) subsets F_0 and F_1 of $X, \exists\, G, H \in \tau$ such that $F_0 \subseteq G$, $F_1 \subseteq H, G \cap H = \emptyset$.

Definition 2.40 [26] : Let X be a topological space. Then X is $\boldsymbol{normal}$ if and only if whenever A is a closed subset in X and $f : A \to \mathbb{R}$ is a continuous function, there is a continuous extension of f to all X, i.e. there is a continuous function $F : X \to \mathbb{R}$ such that $F|A = f$.

Example 2.11.

1. (X,τ_u) is a normal space: Let F_1 and F_2 be any two disjoint closed sets in (X,τ_u). $F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq X - F_1$. Thus for each $x \in F_1, \exists\, r > 0$ such that $(x - r,\ x + r) \subseteq X - F_2$ (since $X - F_2$ is an open set in X). Hence $\left(x - \frac{r}{2}, x + \frac{r}{2}\right) \cap F_2 = \emptyset$.
Let $G = \cup_{x \in F_1}\left(x - \frac{r}{2}, x + \frac{r}{2}\right)$. Then $G \in \tau_u$ and $F_1 \subseteq G$ **----(1).**
Similarly for each $x \in F_2, \exists\, r > 0$ such that $(x - r,\ x + r) \subseteq X - F_1$. Hence $\left(x - \frac{r}{2}, x + \frac{r}{2}\right) \cap F_1 = \emptyset$.
Let $H = \cup_{x \in F_2}\left(x - \frac{r}{2}, x + \frac{r}{2}\right)$. Then $H \in \tau_u$ and $F_2 \subseteq H$ **-----(2)**
Only to prove that $G \cap H = \emptyset$.
Let $a \in G \cap H$. $a \in G \Rightarrow a \in \left(x - \frac{r}{2},\ x + \frac{r}{2}\right)$, , for some $x \in F_1$. $a \in H \Rightarrow a \in \left(y - \frac{\in}{2},\ y + \frac{\in}{2}\right)$, , for some $y \in F_2$. $|x - a| < \frac{r}{2}$ and $|y - a| < \frac{\in}{2}$. Hence $|x - y| = |x - a + y - a| \leq |x - a| + |y - a|$.
As r and $\in$. are real numbers, they are comparable.
<u>Case 1: Let $r < \in$:</u>
Then$|x - y| < \in$ Hence $x \in (y - \in, y + \in)$ By the choice of $\in$. $(y - \in,\ y + \in) \subseteq X - F_1$ Hence $x \in (y - \in,\ y + \in) \Rightarrow x \in X - F_1 \Rightarrow x \notin F_1$; which is contradiction.
<u>Case 2. Let $\in < r$:</u>
Then$|x - y| < r$ Hence $y \in (x - r, x + r)$ By the choice of r. $(x - r, x + r) \subseteq X - F_2$, will imply $x \in X - F_2$ i.e $y \notin F_2$; which is contradiction.
Hence $G \cap H = \emptyset$.
Thus given two disjoint closed sets F_1 and F_2 in $\mathbb{R}$, two disjoint open sets G and H such that $F_1 \subseteq G$ and $F_2 \subseteq H$. Hence (X,τ_u) is a normal space.
2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ Then(X,τ) is a topological space. The Family of closed sets κ is given by $\kappa = \{\emptyset, \{b, c\}, \{a, c\}, \{a\}, X\}$.
Each pair of disjoint closed sets contains $\emptyset$. Hence the space is (X,τ) a normal space.

Lemma 2.4 [17] : $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$.
Proof. $(T_4 \Rightarrow T_{3\frac{1}{2}})$. Let X be T_4 . Let $x \in X$ and $C \subseteq X$ be closed. Since X



is T_1 $\{x\}$ is closed. Hence by Urysohn's Lemma there exists a continuous function $f : X \to [0,1]$ where $f(x) = 0$ and $f(C) = 1$. Therefore X is completely regular.

Theorem 2.27 [23] : A compact Hausdorff space is normal.
Proof. Suppose X is compact Hausdorff. We will first show that X is regular. For this, suppose C is a closed subset and $x \notin C$. Since X is Hausdorff, for any point $y \in C$ there are open sets U_y and V_y with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. Since C is closed, it is compact, and the sets V_y cover it. Thus there are points $y_1, \dots, y_n$ so that $C \subset V_{y1}, \cup \dots \cup V_{yn}$, If we put $U = U_{y1} \cap \dots \cap U_{yn}$ and $V = V_{y1}, \cup \dots \cup V_{yn}$ then $x \in C, C \subset V$, and $U \cap V = \emptyset$ as desired. The remainder of the proof goes exactly the same way with C playing the role of x and the other closed set playing the role of C.

- **Completely Normal Space or T_5 :**

Definition 2.41 [26] : A topological space is called completely normal if every subspace of X is normal. A topological space is $completely\ normal$ if and only if whenever A and B are subsets in X with $A \cap B = A \cap B = \emptyset$, then there are disjoint open subsets $U \supseteq A$ and $V \supseteq B$. (Hint : To do necessity, consider the subspace $X \setminus (A \cap B)$ which contains both A and B and in which A and B have disjoint closures. Sufficiency is easy.)
Definition 2.42 [26] :
(1) Arbitrary subspaces of a normal (resp. $\mathbf{T_4}$) spaces need not be normal (resp.$\mathbf{T_4}$). But every closed subspace of a normal (resp. $\mathbf{T_4}$) space is normal (resp. $\mathbf{T_4}$) If every subspace of a topological space is normal, then X is said to be $completely\ normal$.
(2) The closed continuous image of a normal (resp.$\mathbf{T_4}$) space is normal (resp. $\mathbf{T_4}$).

- **Urysohn's Characterisation of Normality - Urysohn's Lemma or $T_{2\frac{1}{2}}$:**

Definition 2.43 [26] : Let X be a topological space. Then X is $normal$ if and only if whenever A and B are closed subsets in X, there is a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. In particular, every T_4 space is Tychonoff. (Remark : Such a function is called a Urysohn function . Note that it is nowhere claimed, nor, is it true, that f is 0 only on A and 1 only on B. It other words we merely claim that $A \subseteq f^{-1}(0)$ and $\subseteq f^{-1}(1)$. There may be points outside $A \cup B$ at which f is either 0 or 1).

- **Tietze's Characterisation of Normality - Tietze's** (also known as the Tietze–Urysohn–Brouwer extension theorem states that continuous functions on a closed subset of a normal topological space can be extended to the entire space, preserving boundedness if necessary.)

2.5 Mildly Normal Space

Definition 2.44 [14]: A topological space $\boldsymbol{X}$ is called $\boldsymbol{mildly\ normal}$ (**called also** $\boldsymbol{k-normal}$) if for any two disjoint regularly closed subsets $\boldsymbol{A}$ and B of $\boldsymbol{X}$, there exist two open disjoint subsets $\boldsymbol{U}$ and $\boldsymbol{V}$ of $\boldsymbol{X}$ such that $\boldsymbol{A} \subseteq \boldsymbol{U}$ and $\boldsymbol{B} \subseteq \boldsymbol{V}$.

Definition 2.45 [28] : A topological space $\boldsymbol{X}$ is called $\boldsymbol{nearly\ \ compact}$ if, for each open cover $\boldsymbol{u}$ of $\boldsymbol{X}$, there exists a finite subcollection $\boldsymbol{r} \subset \boldsymbol{u}$ such that $\cup\ \{\boldsymbol{Int}(\boldsymbol{Cl}(\boldsymbol{V})) \mid \boldsymbol{V} \in \boldsymbol{r}\} = \boldsymbol{X}$.

- **Almost Continuous in Topology**

Definition 2.46 [29] : A mapping $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is said to be $\boldsymbol{almost-open}$ if the image of every $\boldsymbol{regular-open}$ subset of $\boldsymbol{X}$ is an open subset of $\boldsymbol{Y}$.

Definition 2.47 [29] : A mapping $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is said to be $\boldsymbol{almost-closed}$ if the image of every $\boldsymbol{regular-closed}$ subset of $\boldsymbol{X}$ is an closed subset of $\boldsymbol{Y}$.

Definition 2.48 [30] : A function $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is called $\boldsymbol{almost\ \ continuous}$ if the inverse images of regularly open sets of $\boldsymbol{Y}$ are open in $\boldsymbol{X}$.

Definition 2.49 [16] : A function $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is called an $\boldsymbol{almost\ continuous}$ if for each $\boldsymbol{x} \in \boldsymbol{X}$ and for each regular open set $\boldsymbol{V}$ containing $\boldsymbol{f}(\boldsymbol{x})$, there exists an open set $\boldsymbol{U}$ containing $\boldsymbol{x}$ such that $\boldsymbol{f}(\boldsymbol{U}) \subseteq \boldsymbol{V}$.

Lemma 2.5 [31] : For a mapping $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ following are equivalent :
(1) $\boldsymbol{f}$ is almost continuous
(2) Inverse image of each regular open subset of $\boldsymbol{Y}$ is open subset of $\boldsymbol{X}$.
(3) Inverse image of each regular closed subset of $\boldsymbol{Y}$ is closed subset of $\boldsymbol{X}$.

Theorem 2.28 [29] : Let $\boldsymbol{f}$ map $\boldsymbol{X}$ into $\boldsymbol{Y}$ and let $\boldsymbol{x}$ be a point of $\boldsymbol{X}$. If there exists a neighbourhood $\boldsymbol{N}$ of $\boldsymbol{x}$ such that the restriction of $\boldsymbol{f}$ to $\boldsymbol{N}$ is almost-continuous at $\boldsymbol{x}$, then $\boldsymbol{f}$ is almost-continuous at $\boldsymbol{x}$.
Proof. Let $\boldsymbol{U}$ be any regularly-open set containing $\boldsymbol{f}(\boldsymbol{x})$. Since $\boldsymbol{f}/\boldsymbol{N}$ is almost continuous at $\boldsymbol{x}$, therefore, there is an open set $\boldsymbol{V_1}$ such that $\boldsymbol{x} \in \boldsymbol{N} \cap \boldsymbol{V_1}$ and $\boldsymbol{f}(\boldsymbol{N} \cap \boldsymbol{V_1}) \subset \boldsymbol{U}$. The result now follows from the fact that $\boldsymbol{N} \cap \boldsymbol{V_1}$ is a neighbourhood of **x**.

Definition 2.50 [29] : A mapping $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is said to be $\boldsymbol{almost-continuous}$ at a point $\in \boldsymbol{X}$, if for neighbourhood $\boldsymbol{M}$ of $\boldsymbol{f}(\boldsymbol{x})$ there is a neighbourhood $\boldsymbol{N}$ of $\boldsymbol{x}$ such that $\boldsymbol{f}(\boldsymbol{N}) \subset \boldsymbol{M^{-0}}$. It is easy to see that the neighbourhood $\boldsymbol{M}$ and $\boldsymbol{N}$ can be replaced by open neighbourhood.

Definition 2.51 [32] : The function $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is almost continuous at $\boldsymbol{x_0} \in \boldsymbol{X}$ if and only if for each open $\boldsymbol{V} \subset \boldsymbol{Y}$ containing $\boldsymbol{f}(\boldsymbol{x_o}), \boldsymbol{Cl}(\boldsymbol{f^{-1}}(\boldsymbol{V}))$ is a neighborhood of $\boldsymbol{x_0}$. Iff is almost continuous at each point of $\boldsymbol{X}$, then $\boldsymbol{f}$ is called almost continuous.

Definition 2.52 [32] : The function $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is almost continuous if and only if given any open set $\boldsymbol{W}$ containing the graph of $\boldsymbol{f}$ **,** there exists a continuous $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ such that the graph of g is a subset of $\boldsymbol{W}$**.**

Corollary 2.3 [33] : A function $f : X \to Y$ (almost) continuous if and only if $f(Cl\, U) \subseteq Cl\, f(U)$ for each (open) subset U of X

Lemma 2.6 [32] : Let $f : X \to Y$ be a.c. at $x_0 \in X$ where X is T_1 and $x_0 \in X$ is a limit point of X. Thenfor everypair of open sets $U \subset X$ and $V \subset Y$ containing x_0 and $f(x_0)$, respectively, there exists an $x \in U \setminus \{x_0\}$ such that $f(X) \in V$.
Proof. Assume the conclusion false. Then there exist open sets U_1 and V_1containing x_0 and $f(x_0)$, respectively, such that $U_1 \cap f^{-1}(V_1) = \{x_0\}$. Thus no point of U_1 is a limit point of $f^{-1}(V_1)$ in the T_1 space X. Therefore $U_1 \cap Cl\left(f^{-1}(V_1)\right) = \{x_0\}$ which contradicts the hypothesis that f is a.c. at x_0 .

Theorem 2.29 [29] : If f is an open continuous mapping of X onto Y and if g is a mapping of Y into Z, then gof is almost-continuous iff g is almost-continuous.
Proof. First, let gof be almost-continuous. Let A be a regularly-open subset of Z. Since gof is almost-continuous, therefore $(gof)^{-1}(A)$ is open, that is, $f^{-1}(g^{-1}(A))$ is open. Also, f is open. Therefore $f[f^{-1}(g^{-1}(A))]$ is open, that is, $g^{-1}(A)$ is open and consequently g is almost continuous. Now, let g be almost-continuous and let S be any regularly-open subset of Z. Then $g^{-1}(A)$ is an open subset of Y. Since f is continuous, therefore $f^{-1}(g^{-1}(A))$ is an open subset of X. **(i.e.)** $(gof)^{-1}(A)$ is an open subset of X. Hence gof is almost-continuous.

Theorem 2.29 [29] : If f is a mapping of X into Y and $X = X_1\ U\ X_2$, where X_1 and X_2, are closed and $f\backslash X_1$ and $f\backslash X_2$ are almost-continuous, then f is almost continuous.
Proof. Let A be a regularly-closed subset of Y. Then, since $f\backslash X_1$ and $f\backslash X_2$ are both almost-continuous, therefore if $(f\backslash X_1)^{-1}(A)$ and $(f\backslash X_2)^{-1}(A)$ are both closed X_1 and X_2 respectively. Since X_1 and X_2 are closed subsets of X, therefore $(f\backslash X_1)^{-1}(A)$ and $(f\backslash X_2)^{-1}(A)$ are also closed subsets of X. Also, $f^{-1}(A) = (f\backslash X_1)^{-1}(A) \cup (f\backslash X_2)^{-1}(A)$. Thus $f^{-1}(A)$ is the union of two closed sets and is therefore closed. Hence f is almost-continuous.

Theorem 2.30 [29] : If f is an almost-continuous, closed mapping of regular space X onto a space Y such that $f^{-1}(y)$ is compact for each point $y \in Y$, then Y is almost-regular.
Proof. Let A be a regularly-closed subset of Y and suppose that $y \notin A$. Then, $f^{-1}(y) \cap f^{-1}(A) = \emptyset$, $f^{-1}(A)$ is closed by the almost continuity of f and $f^{-1}(y)$ is compact. Since X is regular, there exist disjoint open sets G and H such that $f^{-1}(A) \subset G$, $f^{-1}(y) \subset H$. Now, let $P = \{z: f^{-1}(z) \subset G\}$ and $Q_y = \{z : f^{-1}(z) \subset H\}$. Then, $y \in P$, $A \subset Q, P \cap Q = \emptyset$. Also since f is closed, therefore P and Q are open. Hence Y is almost-regular.

3. A Space Properties Under The Mapping

Definition 3.1. Let (X, τ) is almost regular space. If $\forall A \subset X, x \notin A$ **,** such that $A \subset U, x \in V, U \cap V = \emptyset$**.** and A is compact. then (X, τ) is almost compact regular space.

Theorem 3.1. Let $f: X \to Y$ is a mapping from topological space X in to topological space Y. If X is

compact regular space. and $\boldsymbol{f}$ is continuous mapping. Then $\boldsymbol{f}(\boldsymbol{A})$ is almost compact regular space in $\boldsymbol{Y}$.
Proof: Assume $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is a continuous mapping in to $\boldsymbol{Y}$ with $\boldsymbol{f}|\boldsymbol{A} : \boldsymbol{A} \to \boldsymbol{f}(\boldsymbol{A})$ and (by **Definition 2.31**) (by **Theorem 2.21**) then$(\boldsymbol{A}, \boldsymbol{\tau_A}) \cong \left(\boldsymbol{f}(\boldsymbol{A}), \boldsymbol{\tau_{f(A)}}\right)$ such that $\boldsymbol{\tau} = \{\boldsymbol{A_i} \mid \boldsymbol{A_i} \subset \boldsymbol{X}\}$; $\boldsymbol{A_i} \in \boldsymbol{\tau} \Rightarrow \boldsymbol{A_i} \in \boldsymbol{\tau} \Rightarrow \boldsymbol{A}$ is closed ($\boldsymbol{A}$ is regular space) and $\boldsymbol{A} = \cup \boldsymbol{A_i}$, $\boldsymbol{A_i} \in \boldsymbol{\tau_A}$. and (by **Definition Proof 2.16**) Then $\boldsymbol{X}$ is compact regular, and (by **Theorem 2.24**) $(\boldsymbol{X}, \boldsymbol{\tau})$ is said to be almost regular if for each $\boldsymbol{\tau}$ - regularly closed subset $\boldsymbol{A}$ of $\boldsymbol{X}$ and each point $\boldsymbol{x} \notin \boldsymbol{A}$ there are disjoint $\boldsymbol{\tau_s}$ - open sets $\boldsymbol{U}$ and $\boldsymbol{V}$ such that $\boldsymbol{A} \subset \boldsymbol{U}$ and $\boldsymbol{x} \in \boldsymbol{V}$, and $\boldsymbol{U} \cap \boldsymbol{V} = \emptyset \Rightarrow \boldsymbol{U}$ is compact and $\boldsymbol{V}$ also compact such that $\boldsymbol{U} = \cup \boldsymbol{U_i}$; $\boldsymbol{U_i} \in \boldsymbol{\tau_s}$, $\boldsymbol{V} = \cup \boldsymbol{V_i}$; $\boldsymbol{V_i} \in \boldsymbol{\tau_s}$
$\therefore \boldsymbol{A}$ is almost compact regular space in $\boldsymbol{X}$, then $\boldsymbol{f}(\boldsymbol{A})$ is almost compact regular space in $\boldsymbol{Y}$. ■

Definition 3.2. Let $(\boldsymbol{X}, \boldsymbol{\tau})$is almost regular space $\forall \boldsymbol{A} \subset \boldsymbol{X}$, $\boldsymbol{x} \notin \boldsymbol{A}, \boldsymbol{A} \subset \boldsymbol{U}, \boldsymbol{x} \notin \boldsymbol{A}$ but $\boldsymbol{x} \in \boldsymbol{V}$such that $\boldsymbol{U} \cap \boldsymbol{V} = \emptyset$ and $\boldsymbol{F}$ is closed subset in $\boldsymbol{X}$. Then $(\boldsymbol{X}, \boldsymbol{\tau})$is almost completely compact regular space.

Theorem 3.2. Let $\boldsymbol{f}: \boldsymbol{X} \to A^c$ is a mapping from topological space $\boldsymbol{X}$ in to $\boldsymbol{A^c}$ in topological space $\boldsymbol{Y}$. If $\boldsymbol{X}$ is regular compact space. And $\boldsymbol{f}$ is a continuous mapping. Then $\boldsymbol{f}(\boldsymbol{F})$ is almost completely compact regular space in $\boldsymbol{Y}$.
Proof: Assume$\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{A}$ is a continuous mapping in to $\boldsymbol{Y}$ with $\boldsymbol{f}/\boldsymbol{F}: \boldsymbol{F} \to \boldsymbol{f}(\boldsymbol{F})$and (by **Definition 2.31**) (and **Theorem 2.2**) then $(\boldsymbol{A}, \boldsymbol{\tau_F}) \cong (\boldsymbol{f}(\boldsymbol{F}), \boldsymbol{\tau_{f(F)}})$ such that $\boldsymbol{\tau} = \{\boldsymbol{F_i} | \boldsymbol{F_i} \subset \boldsymbol{X}\}$; $\boldsymbol{F_i} \in \boldsymbol{\tau} \Leftrightarrow \boldsymbol{F_i}$ is closed. ($\boldsymbol{F}$ is regular space) and (by **Theorem 2.8.a**) and (by **Definition Proof 2.16**). $\boldsymbol{X}$ is compact regular of closed sets such that $\boldsymbol{F_i} = \cup \boldsymbol{F_i}$, $\boldsymbol{F_i} \in \boldsymbol{\tau_F}$. and (by **Definition 2.37**) if for each closed (non-empty) subset $\boldsymbol{F}$of $\boldsymbol{X}$ and each point $\boldsymbol{x} \in \boldsymbol{X} \backslash \boldsymbol{F}$, $\exists$ $\boldsymbol{continuous}$ $\boldsymbol{f} : \boldsymbol{X} \to [\boldsymbol{0}, \boldsymbol{1}]$ such that $\boldsymbol{f}(\boldsymbol{x}) = \{\boldsymbol{1}\}$ and $\boldsymbol{f}(\boldsymbol{F}) = \boldsymbol{0}$. Then $(\boldsymbol{X}, \boldsymbol{\tau})$ is completely regular. and $\boldsymbol{F} \cap \{\boldsymbol{x}\} = \emptyset$ therefore $\boldsymbol{F}$ is compact such that $\boldsymbol{F} = \cup \boldsymbol{F_i}$; $\boldsymbol{F_i} \in \boldsymbol{\tau}$,$\{\boldsymbol{x}\} = \cup \{\boldsymbol{x_i}\}$, $\boldsymbol{x_i} \in \boldsymbol{X}$.
$\therefore$ $\boldsymbol{F}$ is almost completely compact regular space in $\boldsymbol{X}$, then $\boldsymbol{f}(\boldsymbol{F})$ is almost completely compact regular space in $\boldsymbol{Y}$. ■

Definition 3.3. Let $(\boldsymbol{X}, \boldsymbol{\tau})$is regular space such that if $\forall \boldsymbol{F} \subset \boldsymbol{X}$, $\boldsymbol{x} \notin \boldsymbol{F}, \boldsymbol{H} \subset \boldsymbol{\tau}, \boldsymbol{F} \subset \boldsymbol{H}, \boldsymbol{x} \in \boldsymbol{G}, \boldsymbol{G} \cap \boldsymbol{H} = \emptyset$. and $\boldsymbol{F}$ is normal such that $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ of $\boldsymbol{X}, \boldsymbol{F}_0 \subseteq \boldsymbol{G}, \boldsymbol{F}_1 \subseteq \boldsymbol{H}, \boldsymbol{G} \cap \boldsymbol{H} = \emptyset$. Then $(\boldsymbol{X}, \boldsymbol{\tau})$ is normal regular space.

Theorem 3.3. Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is a mapping from topological space $\boldsymbol{X}$ in to topological space $\boldsymbol{Y}$. If $\boldsymbol{X}$ is regular space. and for any two disjoint closed subsetsone of which is regularly closed subsets. Then $\boldsymbol{f}$ is normal regular space in $\boldsymbol{Y}$.
Proof: Assume $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is a continuous mapping in to $\boldsymbol{Y}$ with $\boldsymbol{f}/\boldsymbol{F}: \boldsymbol{F} \to \boldsymbol{f}(\boldsymbol{F})$ and (by **Definition 2.31**) (and by **Theorem 2.21**) then $(\boldsymbol{F}, \boldsymbol{\tau_F}) \cong (\boldsymbol{f}(\boldsymbol{F}), \boldsymbol{\tau_{f(F)}})$ such that $\boldsymbol{\tau} = \{\boldsymbol{F_i} \mid \boldsymbol{F_i} \subset \boldsymbol{X}\}$; $\boldsymbol{F_i} \in \boldsymbol{\tau} \Leftrightarrow \boldsymbol{F_i}$ is closed. Then $\boldsymbol{X}$ is regular, and (by **Definition 2.9**) and (by **Definition 2.37**) for each two disjoint closed (non-empty) subsets $\boldsymbol{F_0}$ and $\boldsymbol{F_1}$ of $\boldsymbol{X}, \exists$ $\boldsymbol{G}, \boldsymbol{H} \in \boldsymbol{\tau}$ such that $\boldsymbol{F_0} \subseteq \boldsymbol{G}, \boldsymbol{F_1} \subseteq \boldsymbol{H}, \boldsymbol{G} \cap \boldsymbol{H} = \emptyset \Rightarrow \boldsymbol{G}$ is normal compact space and $\boldsymbol{H}$ also normal compact space such that$\boldsymbol{F_0} \subset \boldsymbol{G} = \cup (\boldsymbol{F_{0_i}} \subset \boldsymbol{G_i})$; $\boldsymbol{F_{0_i}} \subset \boldsymbol{G_i} \in \boldsymbol{\tau}$, $\boldsymbol{F_1} \subset \boldsymbol{H} = \cup (\boldsymbol{F_{1_i}} \subset \boldsymbol{H_i})$; $\boldsymbol{F_{1_i}} \subset \boldsymbol{H_i} \in \boldsymbol{\tau}$.therefore $\boldsymbol{F_0}$ and $\boldsymbol{F_1}$ are normal regular in $\boldsymbol{X}$, also $\boldsymbol{f}(\boldsymbol{F_0})$ and $\boldsymbol{f}(\boldsymbol{F_1})$ is normal regular in $\boldsymbol{Y}$. ■

Definition 3.4. Let $(\boldsymbol{X}, \boldsymbol{\tau})$is regular space such that if $\forall \boldsymbol{F} \subset \boldsymbol{X}$, $\boldsymbol{x} \notin \boldsymbol{F}, \boldsymbol{H} \subset \boldsymbol{\tau}, \boldsymbol{F} \subset \boldsymbol{H}, \boldsymbol{x} \in \boldsymbol{G}, \boldsymbol{G} \cap \boldsymbol{H} = \emptyset$**.** and $\boldsymbol{X}$ is compact space such that every open covering of $\boldsymbol{X}$ has a finite subcover.and $\boldsymbol{F}$ is normal such

that F_0 and F_1 of X, $F_0 \subseteq G, F_1 \subseteq H, G \cap H = \emptyset$. Then $(X, \tau)$is normal regular space.

Theorem 3.4. Let $f: X \to Y$ is a mapping from topological space X in to topological space Y. If X is regular compact space. and for any two disjoint closed subsetsone of which is regularly closed subsets. Then f is normal compact regular space in Y.

Proof: Assume $f: X \to Y$ is a continuous mapping in to Y with $f/F: F \to f(F)$ and (by **Definition 2.31**) (and by **Theorem 2.21**) then $(F, \tau_F) \cong (f(F), \tau_{f(F)})$ such that $\tau = \{F_i \mid F_i \subset X\}$; $F_i \in \tau \Leftrightarrow F_i$ is closed. and (by **Definition Proof 2.16**). Then X is compact regular space, and (by **Theorem 2.19**) ,and (by **Definition 2.37**) or each two disjoint closed (non-empty) subsets F_0 and F_1 of $X, \exists\, G, H \in \tau$ such that $F_0 \subseteq G, F_1 \subseteq H, G \cap H = \emptyset \Rightarrow G$ is normal and H also normal such that $F_0 \subset G = \cup\left(F_{0_i} \subset G_i\right)$; $F_{0_i} \subset G_i \in \tau, F_1 \subset H = \cup(F_{1_i} \subset H_i)$; $F_{1_i} \subset H_i \in \tau$.

$\therefore F_0$ and F_1 are normal compact regular space in X, also $f(F_0)$and $f(F_1)$ is normal compact regular space in Y.■

Definition 3.5. Let $(X, \tau)$is almost regular space such that $\forall N \subset X,\ x \notin N$, and $\exists\, C, D \in \tau_X \Rightarrow N_1 \subset C$ and $N_2 \subset D$. and $C \cap D = \emptyset$. andfor each two disjoint closed (non-empty) subsets**.** Then $(X, \tau)$is almost normal compact regular space.

Theorem 3.5. Let $f: X \to Y$ is a mapping from topological space X in to topological space Y. If X is regular compact space. and for any two disjoint closed subsets one of which is regularly closed subsets. Then $f(N)$ is almost normal compact regular space in Y.

Proof: Assume $f: X \to Y$ is a continuous mapping in to Y with $f/N: N \to f(N)$ and (by **Definition 2.31**) (and by **Theorem 2.21**) then $(N, \tau_N) \cong (f(N), \tau_{f(N)})$ such that$\tau = \{N_i \mid N_i \subset X\}$; $N_i \in \tau \Leftrightarrow N_i$ is closed. (N is regular space) and $N = \cup N_i,\ N_i \in \tau_N$. and (by **Theorem 2.8.a**) and (by **Definition Proof 2.16**). Then X is compact regular of closed sets, and (by **Definition 2.37**) if for any two disjoint closed subsets N_1 and N_2 of X one of which is regularly closed, there exist two open disjoint subsets C and D of X such that $N_1 \subset C$ and $N_2 \subset D$. And $C \cap D = \emptyset \Rightarrow C$ is normal and D also normal such that $N_1 \subset C = \cup(N_{1i} \subset C_i)$; $N_i \subset C_i \in \tau, N_2 \subset D = \cup(N_{2i} \subset D_i)$; $N_{2i} \subset D_i \in \tau$. and (by **Definition 3.4**) therefore N_1 and N_2 are almost normal regular in X, also $f(N_1)$ and $f(N_2)$ is almost normal compact regular in Y. ■

Definition 3.6. Let $(X, \tau)$is almost compact regular space such that $\forall S \subset X, x \notin S$. And that f is a continuous mapping. Then (X, τ) is almost continuous compact regular space.

Theorem 3.6. Let $f: X \to Y$ is a mapping from topological space X in to topological space Y. If X is regular compact space. and f is a continuous mapping at x_0. Then $f(S)$ is almost continuous compact regular space in Y.

Proof: Assume $f: X \to Y$ is a continuous mapping in to Y with $f/S: S \to f(S)$ and (by **Definition 2.31**) (and by **Theorem 2.21**) then $(S, \tau_S) \cong \left(f(S), \tau_{f(S)}\right)$ such that $\tau = \{S_i \mid S_i \subset X\}$; $S_i \in \tau \Leftrightarrow S_i$ is closed. (S is regular space) and $S = \cup S_i$, $S_i \in \tau_S$. and (by **Definition Proof 2.16**). Then X is compact regular. and (by **Definition 2.21**) Then X is almost continuous compact regular space. And (by **Definition 2.37**) if f is a continuous mapping at x_0 If and only if for every open set V containing $f(x_0)$ in Y. There exist an open set U in X such that $x \in U \subseteq f^{-1}(S)$. then (by **Definition 2.37**) if the inverse images of regularly open

sets (if $\boldsymbol{S} = \boldsymbol{int}(\overline{\boldsymbol{S}}))$(resp. close) of $\boldsymbol{Y}$ are open in $\boldsymbol{X}$ (resp. close) Then $\boldsymbol{f}(\boldsymbol{x_0})$ is continuous mapping. and (by definition continuous compact regular space) Then $\boldsymbol{f}(\boldsymbol{S})$ is almost continuous compact regular space. ■

Definition 3.7. Let $(\boldsymbol{X}, \boldsymbol{\tau})$is almost regular space such that $\forall \boldsymbol{M} \subset \boldsymbol{X}, \boldsymbol{x} \notin \boldsymbol{M}$. And for any two disjoint regularly closed subsets. Then $(\boldsymbol{X}, \boldsymbol{\tau})$ is almost mildly normal compact regular space.

Theorem 3.7. Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is a mapping from topological space $\boldsymbol{X}$ in to topological space $\boldsymbol{Y}$. If $\boldsymbol{X}$ is regular compact space. and for any two disjoint regularly closed subsets. Then $\boldsymbol{f}(\boldsymbol{M})$ is almost mildly normal compact regular space in $\boldsymbol{Y}$.

Proof: Assume$\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ is a continuous mapping in to $\boldsymbol{Y}$ with $\boldsymbol{f} : \boldsymbol{M} \to \boldsymbol{f}(\boldsymbol{M})$ and (by **Definition 2.31**) (and by **Theorem 2.21**) then $(\boldsymbol{M}, \boldsymbol{\tau_M}) \cong \left(\boldsymbol{f}(\boldsymbol{M}), \boldsymbol{\tau_{f(M)}}\right)$ such that $\boldsymbol{\tau} = \{\boldsymbol{M_i} | \ \boldsymbol{M_i} \subset \boldsymbol{X}\}$; $\boldsymbol{M_i} \in \boldsymbol{\tau} \Leftrightarrow \boldsymbol{M_i}$ is closed. ($\boldsymbol{M}$ is regular space) and $\boldsymbol{M} = \cup \boldsymbol{M_i}, \ \boldsymbol{M_i} \in \boldsymbol{\tau}_{\mathbf{M}}$. and (by **Theorem 2.17.a**) and (by **Definition Proof 2.6**) . Then $\boldsymbol{X}$ is compact regular of closed sets, and (by **Definition 2.43**) if for any two disjoint regularly closed subsets $\boldsymbol{M_1}$ and $\boldsymbol{M_2}$ of $\boldsymbol{X}$, there exist two open disjoint subsets $\boldsymbol{U}$ and $\boldsymbol{V}$ of $\boldsymbol{X}$ such that $\boldsymbol{M_1} \subseteq \boldsymbol{U}$ and $\boldsymbol{M_2} \subseteq \boldsymbol{V}$ and $\boldsymbol{U} \cap \boldsymbol{V} = \emptyset \Rightarrow \boldsymbol{U}$ and $\boldsymbol{V}$ are mildly normal such that $\boldsymbol{M_1} \subset \mathbf{U} = \cup (\boldsymbol{M_{1i}} \subset \boldsymbol{U_i})$; $\boldsymbol{M_{1i}} \subset \boldsymbol{U_i} \in \boldsymbol{\tau}, \boldsymbol{M_2} \subset \boldsymbol{V} = \cup (\boldsymbol{M_{2i}} \subset \boldsymbol{V_i})$;$\boldsymbol{M_{2i}} \subset \ \boldsymbol{V_i} \in \ \boldsymbol{\tau}$.

$\therefore \boldsymbol{M_1}$ and $\boldsymbol{M_2}$ are almost mildly normal compact regular space in $\boldsymbol{X}$, also $\boldsymbol{f}(\boldsymbol{M_1})$ and $\boldsymbol{f}(\boldsymbol{M_2})$ are almost mildly normal compact regular space in **Y**. ■

4 . Conclussion

We have already encountered the notion of the weak topology on a given set such that a family of functions is continuous (cf. Definition **3.1**, **3.2**, **3.3**, **3.4**, **3.5**, **3.6**, **3.7**). The weakly open sets are precisely the class of all arbitrary unions of finite intersections of sets of form $\boldsymbol{f^{-1}}$ $(\boldsymbol{U})$ where $\boldsymbol{f} \ \in \boldsymbol{V}^*$ and $\boldsymbol{U}$ is an open set in $\mathbb{R}$ (or $\mathbb{C}$, in the case of complex Banach spaces).

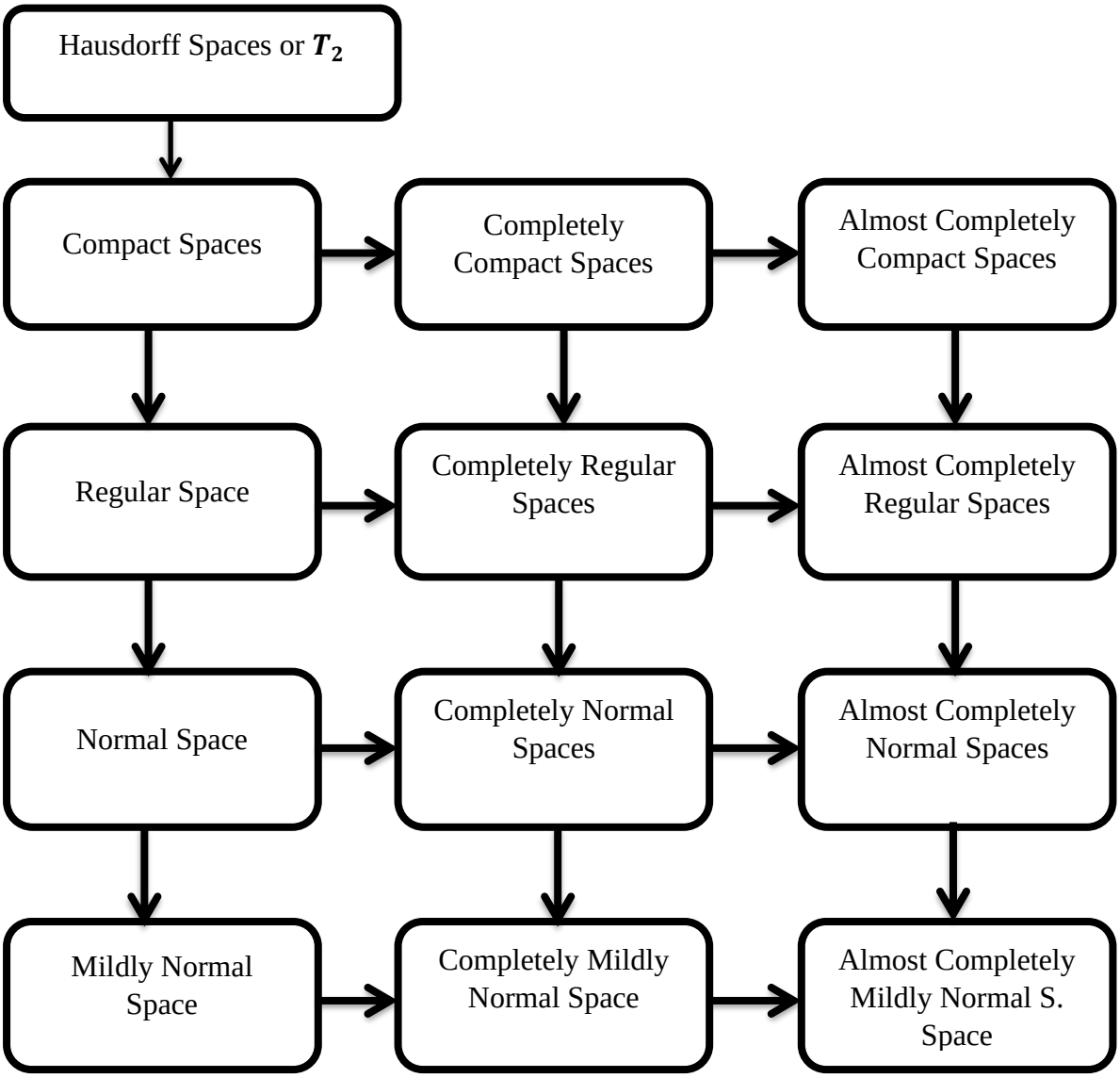
4.1 Future Studies

In the end, we suggest some problems for future studies.

1. We can use the concept of pseudocompact to define a new continuous topological space.

2. Another concept can be used to infer a new space, which is the continuum.

3. Shrinkability and retractability can be used together to define the continuous recoil area, when a lid closed with and retractable, is contractible."


Hausdorff Spaces or T_2
Compact Spaces
Completely Compact Spaces
Almost Completely Compact Spaces
Regular Space
Completely Regular Spaces
Almost Completely Regular Spaces
Normal Space
Completely Normal Spaces
Almost Completely Normal Spaces
Mildly Normal Space
Completely Mildly Normal Space
Almost Completely Mildly Normal S. Space

Reference

[1] N. Palaniappan, K. Chandrasekhara Rao, Regular generalized closed sets, Kyungpook Math. J. 33(2) (1993), 211–219.
[2] M. K. Singal, S. P. Arya, On almost-regular spaces, Glasnik Mat. 4(24) (1969), 89–99.
[3] M. K. Singal, S. P. Arya, Almost normal, almost completely regular spaces, Glasnik Mat. 5(25) (1970), 141–152.
[4] Møller, J. M. (2005). General topology. Matematisk Institut, Universitetsparken, 5.

[5] Bedgood, D. R. (1964). Properties of fixed point spaces in point set topology (Doctoral dissertation, Oklahoma State University).
[6] Leinster, T. General Topology.
[7] J. R. Munkres,Lecture Notes on Topology , November 21st 2018.
[8] S.M.Yahya, Introductory set topology, University of Karachi. 1980. 3rdedition. Printed. Published by the times KarachinSadar Pakistan.
[9] Károlyi, K. (2018). Introductory to Set Theory. Budapest: Eotvos Lorand University.
[10] Freiwald, R. C. (2014). An introduction to set theory and topology. Washington University in St. Louis.
[11] McCluskey, A., & McMaster, B. (2014). Undergraduate topology: a working textbook. OUP Oxford.
[12] Bourbaki, N. (2013). General Topology: Chapters 1–4 (Vol. 18). Springer Science & Business Media.
[13] Habib, A. (2011). Mathematical Sciences Foundation New Delhi September 6, 2011.
[14] Kalantan, L. N., & Allahabi, F. (2008). On almost normality. Demonstratio Mathematica, 41(4), 961-968.
[15] Chapter II Separation Properties.. Pure and Applied Mathematics, (1964).
[16] Nasser, F. M. (2016). on minimal and maximal regular open sets. on minimal and maximal regular open sets.
[17] Miller, J. E. (2013). Finite Topological Spaces.
[18] Dugundji, J. (1966). Topology, Series in advanced mathematics. Boston: Allyn and Bacon, Inc, 16, 447.
[19] Cammaroto, F. (1986). Almost-homeomorphisms and aumosttopological properties. Revista Colombiana de Matemáticas, 20(1-2), 57-77.
[20] H. Maki, J. Umehara, T. Noiri, Every topological space is pre-T1/2, Mem. Fac. Sci. Kochi Univ. Ser. A Math. 17 (1996), 33–42.
[21] Bell, J. (2014). The Stone-ˇCech compactification of Tychonoff spaces.
[22] naik, v. topology: the journey into separation axioms.
[23] Bredon, G. E. (2013). Topology and geometry (Vol. 139). Springer Science & Business Media.
[24] Introduction to Topology, , by Math 490 topology students at the University of Michigan in the Winter 2007 semester.
[25] Hammack, R. H. (2013). Book of proof. Richard Hammack.
[26] Prof. D. P.Patil, Separation Axioms, Indian Institute of Science, Bangalore, November 29, 2004.
[27] Møller, J. M. (2005). General topology. Matematisk Institut, Universitetsparken, 5.
[28] Bagchi, K. B., Mukharjee, A., & Paul, M. (2018). A new approach to nearly compact spaces. Acta et Commentationes Universitatis Tartuensis de Mathematica, 22(1), 137-147.
[29] Singal, M. K., & Singal, A. R. (1968). Almost-continuous mappings.

[30] Ergun, N. (1980). On nearly paracompact spaces. İstanbul University Science Faculty. The Journal Of Mathematics, Physics and Astronomy, 45, 65-87.
[31] Malghan, S. R., & Navalagi, G. B. (1990). Almost p-regular, p-completely regular and almost-p-completely regular spaces. Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, 317-326.
[32] Long, P. E., & McGehee, E. E. (1970). Properties of almost continuous functions. Proceedings of the American Mathematical Society, 24(1), 175-180.
[33] Rose, D. A. (1984). Weak continuity and almost continuity. International Journal of Mathematics and Mathematical Sciences, 7(2), 311-318.