Turkish Online Journal of Qualitative Inquiry (TOJQI)
Volume 12, Issue 7, July 2021: 6311- 6321

Research Article

Observations on the Negative Pell Equation $y^2 = 10x^2 - 54$

[1]D. Maheshwari, 2S. Devibala, [3]M. A. Gopalan
[1,3]Department of Mathematics, Shrimati Indira Gandhi College, Trichy 620002
[2]Department of Mathematics, Sri Meenakshi Govt. Arts College for Women (A), Madurai
matmahes@gmail.com, devibala27@yahoo.com, mayilgopalan@gmail.com

ABSTRACT

The binary quadratic Diophantine equation represented by the negative pellian $y^2 = 10x^2 - 54$ is analysed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained some second order Ramanujan numbers and solutions of other choices of hyperbolas, parabolas.

Keywords: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.

2010 Mathematics subject classification 11D09

INTRODUCTION:

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions where D takes different integral values[1-4]. For an extensive review of various problems, one may refer [5-10]. In this communication, yet another interesting equation given by $y^2 = 10x^2 - 54$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

METHOD OF ANALYSIS:

The negative pell equation representing hyperbola under consideration is

$$y^2 = 10x^2 - 54 \quad (1)$$

The smallest positive integer solutions of (1) are, $x_0 = 3, y_0 = 6.$

Consider the pellian equation $y^2 = 10x^2 + 1$ (2)

The initial solutions of (2) are $\tilde{x}_0 = 6, \tilde{y}_0 = 19.$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (2) is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{10}} g_n, \tilde{y}_n = \frac{1}{2} f_n \quad \text{where } f_n = \left(19 + 6\sqrt{10}\right)^{n+1} + \left(19 - 6\sqrt{10}\right)^{n+1}$$

$$g_n = \left(19 + 6\sqrt{10}\right)^{n+1} - \left(19 - 6\sqrt{10}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by $x_{n+1} = \frac{3}{2} f_n + \frac{3}{\sqrt{10}} g_n$

$y_{n+1} = 3f_n + \frac{15}{\sqrt{10}} g_n$.

The recurrence relations satisfied by the solutions x and y are given by

$x_{n+3} - 38x_{n+2} + x_{n+1} = 0$

$y_{n+3} - 38y_{n+2} + y_{n+1} = 0$.

Some numerical values satisfying (1) are given in the table below:

Table 1: Numerical values

n	x_n	y_n
0	3	6
1	93	294
2	3531	11166
3	134085	424014

From the above table 1, we observe some interesting relations among the solutions which are presented below:

- x_n is always odd and y_n is always even.
- One can generate second order Ramanujan numbers choosing x and y values suitably. A few illustrations given below.

Illustration 1:

$x_1 = 93$

= 1 * 93 = 3 * 31

$= 47^2 - 46^2 = 17^2 - 14^2$

$\Rightarrow 47^2 + 14^2 = 17^2 + 46^2 = 2405$

Thus, 2405 is a second order Ramanujan number.

Illustration 2:

$x_2 = 3531$

= 3531 * 1 = 3 * 1177 = 11 * 321 = 33 * 107

$= 1766^2 - 1765^2 = 590^2 - 587^2 = 166^2 - 155^2 = 70^2 - 37^2$

Now,

$1766^2 - 1765^2 = 590^2 - 587^2$

$$\Rightarrow 1766^2 + 587^2 = 590^2 + 1765^2 = 3463325$$

$$1766^2 - 1765^2 = 166^2 - 155^2$$

$$\Rightarrow 1766^2 + 155^2 = 166^2 + 1765^2 = 3142781$$

$$1766^2 - 1765^2 = 70^2 - 37^2$$

$$\Rightarrow 1766^2 + 37^2 = 70^2 + 1765^2 = 3120125$$

$$590^2 - 587^2 = 166^2 - 155^2$$

$$\Rightarrow 590^2 + 155^2 = 166^2 + 587^2 = 372125$$

$$590^2 - 587^2 = 70^2 - 37^2$$

$$\Rightarrow 590^2 + 37^2 = 70^2 + 587^2 = 349469$$

$$166^2 - 155^2 = 70^2 - 37^2$$

$$\Rightarrow 166^2 + 37^2 = 70^2 + 155^2 = 28925$$

Also,

$$3531 * 1 = 3 * 1177$$
$$\Rightarrow (3531+1)^2 + (3-1177)^2 = (3531-1)^2 + (3+1177)^2 = 13853300$$

$$3531 * 1 = 11 * 321$$
$$\Rightarrow (3531+1)^2 + (11-321)^2 = (3531-1)^2 + (11+321)^2 = 12571124$$

$$3531 * 1 = 33 * 107$$
$$\Rightarrow (3531+1)^2 + (33-107)^2 = (3531-1)^2 + (33+107)^2 = 12480500$$

$$3 * 1177 = 11 * 321$$
$$\Rightarrow (3+1177)^2 + (11-321)^2 = (3-1177)^2 + (11+321)^2 = 1488500$$

$$3 * 1177 = 33 * 107$$
$$\Rightarrow (3+1177)^2 + (33-107)^2 = (3-1177)^2 + (33+107)^2 = 1397876$$

$$11 * 321 = 33 * 107$$
$$\Rightarrow (11+321)^2 + (33-107)^2 = (11-321)^2 + (33+107)^2 = 115700$$

Thus, 3463325, 3142781, 3120125, 372125, 349469, 28925, 13853300, 12571124, 12480500, 1488500, 1397876 and 115700 represent second order Ramanujan numbers with base numbers as real integers.

Illustration 3:

$x_2 = 3531$

$= 3531 * 1 = 3 * 1177 = 11 * 321 = 33 * 107$

Now,

$3531*1 = 3*1177$

$\Rightarrow (3531+i)^2 + (3-i1177)^2 = (3531-i)^2 + (3+i1177)^2 = 11082640$

Also,

$3531*1 = 3*1177$

$\Rightarrow (i3531+1)^2 + (i3-1177)^2 = (i3531-1)^2 + (i3+1177)^2 = -11082640$

$3531*1 = 11*321$

$\Rightarrow (3531+i)^2 + (11-i321)^2 = (3531-i)^2 + (11+i321)^2 = 12365040$

Also,

$3531*1 = 11*321$

$\Rightarrow (i3531+1)^2 + (i11-321)^2 = (i3531-1)^2 + (i11+321)^2 = -12365040$

$3531*1 = 33*107$

$\Rightarrow (3531+i)^2 + (33-i107)^2 = (3531-i)^2 + (33+i107)^2 = 12457600$

Also,

$3531*1 = 33*107$

$\Rightarrow (i3531+1)^2 + (i33-107)^2 = (i3531-1)^2 + (i33+107)^2 = -12457600$

$3*1177 = 11*321$

$\Rightarrow (3+i1177)^2 + (11-i321)^2 = (3-i1177)^2 + (11+i321)^2 = -1488240$

Also,

$3*1177 = 11*321$

$\Rightarrow (3i+1177)^2 + (11i-321)^2 = (3i-1177)^2 + (11i+321)^2 = 1488240$

$3*1177 = 33*107$

$\Rightarrow (3+i1177)^2 + (33-i107)^2 = (3-i1177)^2 + (33+i107)^2 = -1395680$

Also,

$3*1177 = 33*107$

$\Rightarrow (3i+1177)^2 + (33i-107)^2 = (3i-1177)^2 + (33i+107)^2 = 1395680$

$11*321 = 33*107$

$\Rightarrow (11+i321)^2 + (33-i107)^2 = (11-i321)^2 + (33+i107)^2 = -113280$

Also,

$$11*321=33*107$$

$$\Rightarrow (11i+321)^2+(33i-107)^2=(11i-321)^2+(33i+107)^2=113280$$

Thus, 11082640, -11082640, 12365040, -12365040, 12457600, -12457600, -1488240, 1488240, -1395680, 1395680, -113280, 113280 represent second order Ramanujan numbers with base numbers as Gaussian integers.

Each of the following expressions is a nasty number

1. $\frac{2}{9}[49x_{2n+2}-x_{2n+3}+54]$

2. $\frac{1}{171}[1861x_{2n+2}-x_{2n+4}+2052]$

3. $\frac{4}{3}[5x_{2n+2}-y_{2n+2}+9]$

4. $\frac{2}{57}[310x_{2n+2}-2y_{2n+3}+342]$

5. $\frac{2}{2163}[11770x_{2n+2}-2y_{2n+4}+12978]$

6. $\frac{2}{9}[1861x_{2n+3}-49x_{2n+4}+54]$

7. $\frac{2}{57}[10x_{2n+3}-98y_{2n+2}+342]$

8. $\frac{2}{3}[310x_{2n+3}-98y_{2n+3}+18]$

9. $\frac{2}{57}[11770x_{2n+3}-98y_{2n+4}+342]$

10. $\frac{2}{2163}[10x_{2n+4}-3722y_{2n+2}+12978]$

11. $\frac{2}{57}[310x_{2n+4}-3722y_{2n+3}+342]$

12. $\frac{2}{3}[11770x_{2n+4}-3722y_{2n+4}+18]$

13. $\frac{1}{9}[y_{2n+3}-31y_{2n+2}+108]$

14. $\frac{1}{342}[y_{2n+4}-1177y_{2n+2}+4104]$

15. $\frac{1}{9}[31y_{2n+4} - 1177y_{2n+3} + 108]$

Each of the following expressions is a cubical integer

1. $\frac{1}{27}[49x_{3n+3} - x_{3n+4} + 147x_{n+1} - 3x_{n+2}]$

2. $\frac{1}{1026}[1861x_{3n+3} - x_{3n+5} + 5583x_{n+1} - 3x_{n+3}]$

3. $\frac{2}{9}[5x_{3n+3} - y_{3n+3} + 15x_{n+1} - 3y_{n+1}]$

4. $\frac{2}{171}[155x_{3n+3} - y_{3n+4} + 465x_{n+1} - 3y_{n+2}]$

5. $\frac{2}{6489}[5885x_{3n+3} - y_{3n+5} + 17655x_{n+1} - 3y_{n+3}]$

6. $\frac{1}{27}[1861x_{3n+4} - 49x_{3n+5} + 5583x_{n+2} - 147x_{n+3}]$

7. $\frac{2}{171}[5x_{3n+4} - 49y_{3n+3} + 15x_{n+2} - 147y_{n+1}]$

8. $\frac{2}{9}[155x_{3n+4} - 49y_{3n+4} + 465x_{n+2} - 147y_{n+2}]$

9. $\frac{2}{171}[5885x_{3n+4} - 49y_{3n+5} + 17655x_{n+2} - 147y_{n+3}]$

10. $\frac{2}{6489}[5x_{3n+5} - 1861y_{3n+3} + 15x_{n+3} - 5583y_{n+1}]$

11. $\frac{2}{171}[155x_{3n+5} - 1861y_{3n+4} + 465x_{n+3} - 5583y_{n+2}]$

12. $\frac{2}{9}[5885x_{3n+5} - 1861y_{3n+5} + 17655x_{n+3} - 5583y_{n+3}]$

13. $\frac{1}{54}[y_{3n+4} - 31y_{3n+3} + 3y_{n+2} - 93y_{n+1}]$

14. $\frac{1}{2052}[y_{3n+5} - 1177y_{3n+3} + 3y_{n+3} - 3531y_{n+1}]$

15. $\frac{1}{54}[31y_{3n+5} - 1177y_{3n+4} + 93y_{n+3} - 3531y_{n+2}]$

Each of the following expressions represent a biquadratic integer.

1. $\frac{1}{27}[49x_{4n+4} - x_{4n+5} + 196x_{2n+2} - 4x_{2n+3} + 162]$

2. $\frac{1}{1026}[1861x_{4n+4} - x_{4n+6} + 7444x_{2n+2} - 4x_{2n+4} + 6156]$

3. $\frac{2}{9}[5x_{4n+4} - y_{4n+4} + 20x_{2n+2} - 4y_{2n+2} + 27]$

4. $\frac{2}{171}[155x_{4n+4} - y_{4n+5} + 620x_{2n+2} - 4y_{2n+3} + 513]$

5. $\frac{2}{6489}[5885x_{4n+4} - y_{4n+6} + 23540x_{2n+2} - 4y_{2n+4} + 19467]$

6. $\frac{1}{27}[1861x_{4n+5} - 49x_{4n+6} + 7444x_{2n+3} - 196x_{2n+4} + 162]$

7. $\frac{2}{171}[5x_{4n+5} - 49y_{4n+4} + 20x_{2n+3} - 196y_{2n+2} + 513]$

8. $\frac{2}{9}[155x_{4n+5} - 49y_{4n+5} + 620x_{2n+3} - 196y_{2n+3} + 27]$

9. $\frac{2}{171}[5885x_{4n+5} - 49y_{4n+6} + 23540x_{2n+3} - 196y_{2n+4} + 513]$

10. $\frac{2}{6489}[5x_{4n+6} - 1861y_{4n+4} + 20x_{2n+4} - 7444y_{2n+2} + 19467]$

11. $\frac{2}{171}[155x_{4n+6} - 1861y_{4n+5} + 620x_{2n+4} - 7444y_{2n+3} + 513]$

12. $\frac{2}{9}[5885x_{4n+6} - 1861y_{4n+6} + 23540x_{2n+4} - 7444y_{2n+4} + 27]$

13. $\frac{1}{54}[y_{4n+5} - 31y_{4n+4} + 4y_{2n+3} - 124y_{2n+2} + 324]$

14. $\frac{1}{2052}[y_{4n+6} - 1177y_{4n+4} + 4y_{2n+4} - 4708y_{2n+2} + 12312]$

15. $\frac{1}{54}[31y_{4n+6} - 1177y_{4n+5} + 124y_{2n+4} - 4708y_{2n+3} + 324]$

Each of the following expressions is a quintic integer:

1. $\frac{1}{27}[49x_{5n+5} - x_{5n+6} + 245x_{3n+3} - 5x_{3n+4} + 490x_{n+1} - 10x_{n+2}]$

2. $\frac{1}{1026}[1861x_{5n+5} - x_{5n+7} + 9305x_{3n+3} - 5x_{3n+5} + 18610x_{n+1} - 10x_{n+3}]$

3. $\frac{2}{9}[5x_{5n+5} - y_{5n+5} + 25x_{3n+3} - 5y_{3n+3} + 50x_{n+1} - 10y_{n+1}]$

4. $\frac{2}{171}[155x_{5n+5} - y_{5n+6} + 775x_{3n+3} - 5y_{3n+4} + 1550x_{n+1} - 10y_{n+2}]$

5. $\frac{2}{6489}[5885x_{5n+5} - y_{5n+7} + 29425x_{3n+3} - 5y_{3n+5} + 58850x_{n+1} - 10y_{n+3}]$

6. $\frac{1}{27}[1861x_{5n+6} - 49x_{5n+7} + 9305x_{3n+4} - 245x_{3n+5} + 18610x_{n+2} - 490x_{n+3}]$

7. $\frac{2}{171}[5x_{5n+6} - 49y_{5n+5} + 25x_{3n+4} - 245y_{3n+3} + 50x_{n+2} - 490y_{n+1}]$

8. $\frac{2}{9}[155x_{5n+6} - 49y_{5n+6} + 775x_{3n+4} - 245y_{3n+4} + 1550x_{n+2} - 490y_{n+2}]$

9. $\frac{2}{171}[5885x_{5n+6} - 49y_{5n+7} + 29425x_{3n+4} - 245y_{3n+5} + 58850x_{n+2} - 490y_{n+3}]$

10. $\frac{2}{6489}[5x_{5n+7} - 1861y_{5n+5} + 25x_{3n+5} - 9305y_{3n+3} + 50x_{n+3} - 18610y_{n+1}]$

11. $\frac{2}{171}[155x_{5n+7} - 1861y_{5n+6} + 775x_{3n+5} - 9305y_{3n+4} + 1550x_{n+3} - 18610y_{n+2}]$

12. $\frac{2}{9}[5885x_{5n+7} - 1861y_{5n+7} + 29425x_{3n+5} - 9305y_{3n+5} + 5885x_{n+3} - 18610y_{n+3}]$

13. $\frac{1}{54}[y_{5n+6} - 31y_{5n+5} + 5y_{3n+4} - 155y_{3n+3} + 10y_{n+2} - 310y_{n+1}]$

14. $\frac{1}{2052}[y_{5n+7} - 1177y_{5n+5} + 5y_{3n+5} - 5885y_{3n+3} + 10y_{n+3} - 11770y_{n+1}]$

15. $\frac{1}{54}[31y_{5n+7} - 1177y_{5n+6} + 155y_{3n+5} - 5885y_{3n+4} + 310y_{n+3} - 11770y_{n+2}]$

Relations among the solutions are given below:

- ➢ $x_{n+2} = 6y_{n+1} + 19x_{n+1}$
- ➢ $19x_{n+2} = 6y_{n+2} + x_{n+1}$
- ➢ $721x_{n+2} = 6y_{n+3} + 19x_{n+1}$

- $x_{n+3} = 228y_{n+1} + 721x_{n+1}$
- $x_{n+3} = 12y_{n+2} + x_{n+1}$
- $721x_{n+3} = 228y_{n+3} + x_{n+1}$
- $y_{n+2} = 60x_{n+1} + 19y_{n+1}$
- $y_{n+3} = 2280x_{n+1} + 721y_{n+1}$
- $19y_{n+3} = 60x_{n+1} + 721y_{n+2}$
- $19x_{n+3} = 721x_{n+2} + 6y_{n+1}$
- $x_{n+3} = 6y_{n+2} + 19x_{n+2}$
- $19x_{n+3} = x_{n+2} + 6y_{n+3}$
- $19y_{n+2} = 60x_{n+2} + y_{n+1}$
- $y_{n+3} = 120x_{n+2} + y_{n+1}$
- $y_{n+3} = 60x_{n+2} + 19y_{n+2}$
- $721y_{n+2} = 60x_{n+3} + 19y_{n+1}$
- $721y_{n+3} = 2280x_{n+3} + y_{n+1}$
- $19y_{n+3} = 60x_{n+3} + y_{n+2}$

Remarkable observations:

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below.

Table 2: Hyperbola

S.No	Hyperbola	(X,Y)
1	$10Y^2 - X^2 = 29160$	$(5x_{n+2} - 155x_{n+1}, 49x_{n+1} - x_{n+2})$
2	$2Y^2 - 5X^2 = 8421408$	$(x_{n+3} - 1177x_{n+1}, 1861x_{n+1} - x_{n+3})$
3	$2Y^2 - 5X^2 = 162$	$(y_{n+1} - 2x_{n+1}, 5x_{n+1} - y_{n+1})$
4	$Y^2 - 10X^2 = 116964$	$(y_{n+2} - 98x_{n+1}, 310x_{n+1} - 2y_{n+2})$
5	$Y^2 - 10X^2 = 168428484$	$(y_{n+3} - 3722x_{n+1}, 11770x_{n+1} - 2y_{n+3})$
6	$2Y^2 - 5X^2 = 5832$	$(31x_{n+3} - 1177x_{n+2}, 1861x_{n+2} - 49x_{n+3})$
7	$Y^2 - 10X^2 = 116964$	$(31y_{n+1} - 2x_{n+2}, 10x_{n+2} - 98y_{n+1})$
8	$Y^2 - 10X^2 = 324$	$(31y_{n+2} - 98x_{n+2}, 310x_{n+2} - 98y_{n+2})$

9	$Y^2 - 10X^2 = 116964$	$(31y_{n+3} - 3722x_{n+2}, 11770x_{n+2} - 98y_{n+3})$
10	$Y^2 - 10X^2 = 168428484$	$(1177y_{n+1} - 2x_{n+3}, 10x_{n+3} - 3722y_{n+1})$
11	$Y^2 - 10X^2 = 116964$	$(1177y_{n+2} - 98x_{n+3}, 310x_{n+3} - 3722y_{n+2})$
12	$Y^2 - 10X^2 = 324$	$(1177y_{n+3} - 3722x_{n+3}, 11770x_{n+3} - 3722y_{n+3})$
13	$5Y^2 - 2X^2 = 58320$	$(49y_{n+1} - y_{n+2}, y_{n+2} - 31y_{n+1})$
14	$5Y^2 - 2X^2 = 84214080$	$(1861y_{n+1} - y_{n+3}, y_{n+3} - 1177y_{n+1})$
15	$5Y^2 - 2X^2 = 58320$	$(1861y_{n+2} - 49y_{n+3}, 31y_{n+3} - 1177y_{n+2})$

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below.

Table 3: Parabola

S.No	Parabola	(X,Y)
1	$270Y - X^2 = 29160$	$(5x_{n+2} - 155x_{n+1}, 49x_{2n+2} - x_{2n+3} + 54)$
2	$2052Y - 5X^2 = 8421408$	$(x_{n+3} - 1177x_{n+1}, 1861x_{2n+2} - x_{2n+4} + 2052)$
3	$9Y - 5X^2 = 162$	$(y_{n+1} - 2x_{n+1}, 5x_{2n+2} - y_{2n+2} + 9)$
4	$171Y - 10X^2 = 116964$	$(y_{n+2} - 98x_{n+1}, 310x_{2n+2} - 2y_{2n+3} + 342)$
5	$6489Y - 10X^2 = 168428484$	$(y_{n+3} - 3722x_{n+1}, 11770x_{2n+2} - 2y_{2n+4} + 12978)$
6	$54Y - 5X^2 = 5832$	$(31x_{n+3} - 1177x_{n+2}, 1861x_{2n+3} - 49x_{2n+4} + 54)$
7	$171Y - 10X^2 = 116964$	$(31y_{n+1} - 2x_{n+2}, 10x_{2n+3} - 98y_{2n+2} + 342)$
8	$9Y - 10X^2 = 324$	$(31y_{n+2} - 98x_{n+2}, 310x_{2n+3} - 98y_{2n+3} + 18)$
9	$171Y - 10X^2 = 116964$	$(31y_{n+3} - 3722x_{n+2}, 11770x_{2n+3} - 98y_{2n+4} + 342)$
10	$6489Y - 10X^2 = 168428484$	$(1177y_{n+1} - 2x_{n+3}, 10x_{2n+4} - 3722y_{2n+2} + 12978)$
11	$171Y - 10X^2 = 116964$	$(1177y_{n+2} - 98x_{n+3}, 310x_{2n+4} - 3722y_{2n+3} + 342)$
12	$9Y - 10X^2 = 324$	$(1177y_{n+3} - 3722x_{n+3}, 11770x_{2n+4} - 3722y_{2n+4} + 18)$
13	$135Y - X^2 = 29160$	$(49y_{n+1} - y_{n+2}, y_{2n+3} - 31y_{2n+2} + 108)$
14	$5130Y - X^2 = 42107040$	$(1861y_{n+1} - y_{n+3}, y_{2n+4} - 1177y_{2n+2} + 4104)$
15	$135Y - X^2 = 29160$	$(1861y_{n+2} - 49y_{n+3}, 31y_{2n+4} - 1177y_{2n+3} + 108)$

CONCLUSION

In this paper, we have presented infinitely many integer solutions for the negative Pell equation $y^2 = 10x^2 - 54$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of Pell equations and determine their integer solutions along with suitable properties.

REFERENCES

1. L.E.Dickson, History of theory of number, Chelsa publishing company, vol-2, (1952) New York.

2. L.J.Mordel, Diophantine equations, Academic Press, (1969) New York.
3. S.J.Telang, Number Theory, Tata McGraw Hill Publishing company Limited,(2000) New Delhi.
4. D.M.Burton, Elementary Number Theory, Tata McGraw Hill Publishing company Limited,(2002) New Delhi.
5. M.A.Gopalan,S.vidhyalakshmi,D.Maheswari, Observations on the hyperpola $y^2 = 30x^2 + 1$ IJOER Vol-1,issue 3, (2013),312-314.
6. M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, observations on the Hyperbola $10y^2 - 30x^2 = 13$, Archimedes J.Math.,3(1) (2013) 31-34.
7. R.Suganya,D.Maheswari, ‘’On the negative pellian eqution $y^2 = 110x^2 - 29$,Journal of the mathematics and informatics,11(2017) 63- 71.
8. M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, on the integral solutions of the binary quadratic equation $x^2 = 15y^2 - 11$, scholars journal of Engineering and technology, 2(2A) (2014) 156-158.
9. M.A.Gopalan, S.Vidhyalakshmi, D.Maheswari, ,Observations on the hyperbola $y^2 = 34x^2 + 1$, Bulletin of mathematics and statistics research, Vol-2, Issue 4(2014) 414-417.
10. D.Maheswari and K.Kaviyarasi, On the negative pell $y^2 = 102x^2 - 18$, Global Scientific Journals, 7(4) April (2019) 521-528.