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Research Article

On Mersenne and Mersenne - Lucas Quaternions and Octonions

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ABSTRACT
In this study, we introduce new class of quaternion and octonion numbers associated with the Mersenne numbers. We define Mersenne quaternion and Mersenne-Lucas quaternion, Mersenne octonion and Mersenne-Lucas octonion by using the Mersenne numbers. We obtained generating functions and Binet formulas for the Mersenne and Mersenne-Lucas quaternion and octonion and some of interesting identities of these numbers.

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INTRODUCTION

Quaternions are non-commutative and non-associative number system that extends the complex numbers. Quaternions were first described by Robrigues in 1840, but in 1843, Hamilton developed independently and applied to Mechanics in three dimensional space. In 1886, Lipschitz started the study of integral quaternions whose system was later simplified by Dickson. In 1845, octonions were developed independently by Cayley and Graves. Octonions have eight dimensions twice the number of dimensions of the quaternions of which they are an extension.

In this communication, we introduce new class of quaternion and octonion numbers associated with the Mersenne numbers. We define Mersenne quaternion and Mersenne-Lucas quaternion, Mersenne octonion and Mersenne-Lucas octonion by using the Mersenne numbers. We obtained generating functions and Binet formulas for the Mersenne and Mersenne-Lucas quaternion and octonion and some of interesting identities of these numbers.

Preliminaries

The Mersenne numbers $\{M_n\}$ is defined by the recurrence relation

$M_n = 3M_{n-1} - 2M_{n-2}$, $n \geq 2$ with $M_0 = 0, M_1 = 1$.

The Mersenne- Lucas numbers $\{ML_n\}$ is defined recurrently by

$ML_n = 3ML_{n-1} - 2ML_{n-2}$, $n \geq 2$ with $ML_0 = 2, ML_1 = 3$.

The generating functions for these sequences are

$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1-3x+2x^2}$ and $\sum_{n=0}^{\infty} ML_n x^n = \frac{2-3x}{1-3x+2x^2}$

The Binet formulas for these sequences are $\mathrm{M_n} = \alpha^{n} - \beta^{n}$ and $\mathrm{ML_n} = \alpha^{n} + \beta^{n}$ where $\alpha = 2,\ \beta = 1$ are roots of the characteristic equation $x^2 - 3x + 2 = 0$

A Quaternions $\mathcal{Q}$ can be written as $\mathcal{Q} = a + bi + cj + dk$, where a, b, c, d are reals.

$i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k, jk = -kj = i, ki = -ik = j$

The Mersenne quaternions and Mersenne-Lucas quaternions are defined by

$$\widetilde{\mathrm{M_n}} = \mathrm{M_n} + i\mathrm{M_{n+1}} + j\mathrm{M_{n+2}} + k\mathrm{M_{n+3}} \quad (1.1)$$

$$\widetilde{\mathrm{ML_n}} = \mathrm{ML_n} + i\mathrm{ML_{n+1}} + j\mathrm{ML_{n+2}} + k\mathrm{ML_{n+3}} \quad (1.2)$$

Multiplication table for the basis of $\mathcal{Q}$.

.	0	1	2	3
0	0	1	2	3
1	1	-0	3	-2
2	2	-3	-0	1
3	3	2	-1	-0

A Octonions $\mathcal{O}$ can be written as $\mathcal{O} = \sum_{s=0}^{7} p_s e_s$, where p_s are reals and $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is the standard octonion basis.Multiplication table for the basis of $\mathcal{O}$.

.	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	3	-2	5	-4	-7	6
2	2	-3	-0	1	6	7	-4	-5
3	3	2	-1	-0	7	-6	5	-4
4	4	-5	-6	-7	-0	1	2	3
5	5	4	-7	6	-1	-0	-3	2
6	6	7	4	-5	-2	3	-0	-1
7	7	-6	5	4	-3	-2	1	-0

The Mersenne octonions and Mersenne-Lucas octonions are defined by

$$\widetilde{M_n} = M_n e_0 + M_{n+1} e_1 + M_{n+2} e_2 + M_{n+3} e_3 + M_{n+4} e_4 + M_{n+5} e_5 + M_{n+6} e_6 + M_{n+7} e_7 \quad (1.3)$$

$$\widetilde{ML_n} = ML_n e_0 + ML_{n+1} e_1 + ML_{n+2} e_2 + ML_{n+3} e_3 + M_{Ln+4} e_4 + ML_{n+5} e_5 + ML_{n+6} e_6 + ML_{n+7} e_7 \quad (1.4)$$

$$\widetilde{M_n} \pm \widetilde{ML_n} = \sum_{i=0}^{7} (M_n \pm ML_n) e_i \quad (1.5)$$

The Mersenne and Mersenne-Lucas quaternions:

Theorem 1. The generating functions for Mersenne quaternions and Mersenne-Lucas quaternions are

$\widetilde{f(x)} = \frac{\widetilde{M_0}+(\widetilde{M_1}-3\widetilde{M_0})x}{1-3x+2x^2}$ and $\widetilde{g(x)} = \frac{\widetilde{ML_0}+(\widetilde{ML_1}-3\widetilde{ML_0})x}{1-3x+2x^2}$

Proof. Define $\widetilde{f(x)} = \sum_{i=0}^{\infty} \widetilde{M_\iota} x^i$

$$\widetilde{f(x)} = \widetilde{M_0} + \widetilde{M_1}x + \sum_{i=2}^{\infty} \widetilde{M_\iota}x^i$$

$$-3x\widetilde{f(x)} = -3\widetilde{M_0}x - 3\sum_{i=2}^{\infty} \widetilde{M_{\iota-1}}x^i$$

$$2x^2\widetilde{f(x)} = 2\sum_{i=2}^{\infty} \widetilde{M_{\iota-2}}x^i$$

$$(1-3x+2x^2)\widetilde{f(x)} = \widetilde{M_0} + \widetilde{M_1}x - 3\widetilde{M_0}x + \sum_{i=2}^{\infty}\left(\widetilde{M_\iota} - 3\widetilde{M_{\iota-1}} + 2\widetilde{M_{\iota-2}}\right)x^i$$

$$\therefore \widetilde{f(x)} = \frac{\widetilde{M_0} + (\widetilde{M_1} - 3\widetilde{M_0})x}{1-3x+2x^2}$$

And define $\widetilde{g(x)} = \sum_{i=0}^{\infty} \widetilde{ML_\iota}x^i$

Multiplying this equation by $1, 3x, 2x^2$ respectively and summing these equations, we obtain

$$(1-3x+2x^2)\widetilde{g(x)} = \widetilde{ML_0} + \widetilde{ML_1}x - 3\widetilde{ML_0}x + \sum_{i=2}^{\infty}\left(\widetilde{ML_\iota} - 3\widetilde{ML_{\iota-1}} + 2\widetilde{ML_{\iota-2}}\right)x^i$$

$$\widetilde{g(x)} = \frac{\widetilde{ML_0} + (\widetilde{ML_1} - 3\widetilde{ML_0})x}{1-3x+2x^2}$$

Theorem 2.The Binet formulas for the Mersenne Quaternions and Mersenne-Lucas Quaternions are

$$\widetilde{M_n} = 2^n\tilde{A} - \breve{B} \tag{2.1}$$

and $$\widetilde{ML_n} = 2^n\tilde{A} + \breve{B} \tag{2.2}$$

where $\tilde{A} = 1 + 2i + 2^2j + 2^3k$, $\tilde{B} = 1 + i + j + k$

Proof. $\widetilde{M_n} = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3}$

$= (2^n - 1) + i(2^{n+1} - 1) + j(2^{n+2} - 1) + k(2^{n+3} - 1)$

$= 2^n\tilde{A} - \tilde{B}$, where $\tilde{A} = 1 + 2i + 2^2j + 2^3k$, $\tilde{B} = 1 + i + j + k$

$\widetilde{ML_n} = ML_n + iML_{n+1} + jML_{n+2} + kML_{n+3}$

$= (2^n + 1) + i(2^{n+1} + 1) + j(2^{n+2} + 1) + k(2^{n+3} + 1)$

$= 2^n\tilde{A} + \tilde{B}$, where $\tilde{A} = 1 + 2i + 2^2j + 2^3k$, $\tilde{B} = 1 + i + j + k$

Lemma 1.

$\tilde{A}\tilde{B} = -13 - i + 11j + 7k$ and $\tilde{B}\tilde{A} = -13 + 7i - j + 11k$

Proof. From the definition of $\tilde{A}$ and $\tilde{B}$, and using the multiplication table for the basis of quaternions, we computed these results.

Theorem 3 (Catalan Identity). Let $n, r \in \mathbb{Z}$, then we have

$$\widetilde{M_{n-r}}\widetilde{M_{n+r}} - \widetilde{M_n^2} = -2^{n-r}\left[2^{2r}\tilde{B}\tilde{A} + \tilde{A}\tilde{B}(1 - 2^{r+1})\right]$$

$$\widetilde{ML_{n-r}}\widetilde{ML_{n+r}} - \widetilde{ML_n^2} = 2^{n-r}\left[2^{2r}\tilde{B}\tilde{A} + \tilde{A}\tilde{B}(1 - 2^{r+1})\right]$$

where $\tilde{A} = 1 + 2i + 2^2j + 2^3k$, $\tilde{B} = 1 + i + j + k$

Proof. Using Binet formulas for $\widetilde{M_n}$ and $\widetilde{ML_n}$, we have

$$\begin{aligned}\widetilde{M_{n-r}}\widetilde{M_{n+r}} - \widetilde{M_n^2} &= \left(2^{n-r}\tilde{A} - \tilde{B}\right)\left(2^{n+r}\tilde{A} - \tilde{B}\right) - \left(2^n\tilde{A} - \tilde{B}\right)^2\\ &= 2^{2n}\tilde{A}^2 - 2^{n+r}\tilde{B}\tilde{A} - 2^{n-r}\tilde{A}\tilde{B} + \tilde{B}^2 - 2^{2n}\tilde{A}^2 - \tilde{B}^2 + 2^{n+1}\tilde{A}\tilde{B}\\ &= -2^{n-r}\left[2^{2r}\tilde{B}\tilde{A} + \tilde{A}\tilde{B}(1 - 2^{r+1})\right]\end{aligned}$$

$$\begin{aligned}\widetilde{ML_{n-r}}\widetilde{ML_{n+r}} - \widetilde{ML_n^2} &= \left(2^{n-r}\tilde{A} + \tilde{B}\right)\left(2^{n+r}\tilde{A} + \tilde{B}\right) - \left(2^n\tilde{A} + \tilde{B}\right)^2\\ &= 2^{2n}\tilde{A}^2 + 2^{n+r}\tilde{B}\tilde{A} + 2^{n-r}\tilde{A}\tilde{B} + \tilde{B}^2 - 2^{2n}\tilde{A}^2 - \tilde{B}^2 - 2^{n+1}\tilde{A}\tilde{B}\\ &= 2^{\mathrm{n-r}}\left[2^{2\mathrm{r}}\tilde{\mathrm{B}}\tilde{\mathrm{A}} + \tilde{\mathrm{A}}\tilde{\mathrm{B}}(1 - 2^{\mathrm{r+1}})\right]\end{aligned}$$

Theorem 4 (Cassini Identity). For any integer n, we have

$$\widetilde{M_{n-1}}\widetilde{M_{n+1}} - \widetilde{M_n^2} = -2^{n-1}\left[4\tilde{B}\tilde{A} - 3\tilde{A}\tilde{B}\right]$$

$$\widetilde{ML_{n-1}}\widetilde{ML_{n+1}} - \widetilde{ML_n^2} = 2^{n-1}\left[4\tilde{B}\tilde{A} - 3\tilde{A}\tilde{B}\right]$$

where $\tilde{A} = 1 + 2i + 2^2j + 2^3k$, $\tilde{B} = 1 + i + j + k$

Proof. Since Cassini's identity is a special case of Catalan's identity, we get this result by substituting $\mathrm{r} = 1$ in Catalan's identity.

Theorem 5 (d'Ocagne's Identity). For any integer m, n, we have

$$\widetilde{M_m}\widetilde{M_{n+1}} - \widetilde{M_{m+1}}\widetilde{M_n} = -2^n\tilde{B}\tilde{A} + 2^m\tilde{A}\tilde{B}$$

$$\widetilde{\mathrm{ML_m}}\widetilde{\mathrm{ML_{n+1}}} - \widetilde{\mathrm{ML_{m+1}}}\widetilde{\mathrm{ML_n}} = 2^{\mathrm{n}}\tilde{\mathrm{B}}\tilde{\mathrm{A}} - 2^{\mathrm{m}}\tilde{\mathrm{A}}\tilde{\mathrm{B}}$$

where $\tilde{A} = 1 + 2i + 2^2j + 2^3k$, $\tilde{B} = 1 + i + j + k$

Proof. By using (2.1) and (2.2), we have

$$\begin{aligned}\widetilde{M_m}\widetilde{M_{n+1}} - \widetilde{M_{m+1}}\widetilde{M_n} &= \left(2^m\tilde{A} - \tilde{B}\right)\left(2^{n+1}\tilde{A} - \tilde{B}\right) - \left(2^{m+1}\tilde{A} - \tilde{B}\right)\left(2^n\tilde{A} - \tilde{B}\right)\\ &= 2^{m+n+1}\tilde{A}^2 - 2^{n+1}\tilde{B}\tilde{A} - 2^m\tilde{A}\tilde{B} + \tilde{B}^2 - 2^{m+n+1}\tilde{A}^2 - \tilde{B}^2 + 2^{m+1}\tilde{A}\tilde{B} + 2^n\tilde{B}\tilde{A}\\ &= -2^n\tilde{B}\tilde{A} + 2^m\tilde{A}\tilde{B}\end{aligned}$$

$$\begin{aligned}\widetilde{ML_m}\widetilde{ML_{n+1}} - \widetilde{ML_{m+1}}\widetilde{ML_n} &= \left(2^m\tilde{A} + \tilde{B}\right)\left(2^{n+1}\tilde{A} + \tilde{B}\right) - \left(2^{m+1}\tilde{A} + \tilde{B}\right)\left(2^n\tilde{A} + \tilde{B}\right)\\ &= 2^{m+n+1}\tilde{A}^2 + 2^{n+1}\tilde{B}\tilde{A} + 2^m\tilde{A}\tilde{B} + \tilde{B}^2 - 2^{m+n+1}\tilde{A}^2 - \tilde{B}^2 - 2^{m+1}\tilde{A}\tilde{B} - 2^n\tilde{B}\tilde{A}\\ &= 2^n\tilde{B}\tilde{A} - 2^m\tilde{A}\tilde{B}\end{aligned}$$

Theorem 6. Let $n \geq 1, r \geq 1$ be integers. Then

i. $\widetilde{M_{n+1}} + \widetilde{M_n} = 3(2^n)\tilde{A} - 2\tilde{B}$

ii. $\widetilde{M_{n+1}} - \widetilde{M_n} = 2^n \tilde{A}$

iii. $\widetilde{M_{n+r}} + \widetilde{M_{n-r}} = 2^{n-r}(2^{2r}+1)\tilde{A} - 2\tilde{B}$

iv. $\widetilde{M_{n+r}} - \widetilde{M_{n-r}} = 2^{n-r}(2^{2r}-1)\tilde{A}$

where $\tilde{A} = 1 + 2i + 2^2 j + 2^3 k$, $\tilde{B} = 1 + i + j + k$

Proof. Using (1.1) and the identities, $M_{n+1} + M_n = 3(2^n) - 2$ and $M_{n+1} - M_n = 2^n$ we have

$$\begin{aligned}\widetilde{M_{n+1}} + \widetilde{M_n} &= (M_{n+1}+M_n) + i(M_{n+2}+M_{n+1}) + j(M_{n+3}+M_{n+2}) + k(M_{n+4}+M_{n+3})\\ &= [3(2^n)-2] + i[3(2^{n+1})-2] + j[3(2^{n+2})-2] + k[3(2^{n+3})-2]\\ &= 3(2^n)\tilde{A} - 2\tilde{B}\end{aligned}$$

$$\begin{aligned}\widetilde{M_{n+1}} - \widetilde{M_n} &= (M_{n+1}-M_n) + i(M_{n+2}-M_{n+1}) + j(M_{n+3}-M_{n+2}) + k(M_{n+4}-M_{n+3})\\ &= 2^n + i2^{n+1} + j2^{n+2} + k2^{n+3}\\ &= 2^n\tilde{A}\end{aligned}$$

By defining, $\widetilde{M_{n+r}} = M_{n+r} + iM_{n+r+1} + jM_{n+r+2} + kM_{n+r+3}$,

$\widetilde{M_{n-r}} = M_{n-r} + iM_{n-r+1} + jM_{n-r+2} + kM_{n-r+3}$ and using $M_{n+r} + M_{n-r} = 2^{n-r}(2^{2r}+1) - 2$

$$\begin{aligned}\widetilde{M_{n+r}} + \widetilde{M_{n-r}} &= (M_{n+r}+M_{n-r}) + i(M_{n+r+1}+M_{n-r+1}) + j(M_{n+r+2}+M_{n-r+2})\\ &\quad + k(M_{n+r+3}+M_{n-r+3})\end{aligned}$$

$$\begin{aligned}&= [2^{n-r}(2^{2r}+1)-2] + i[2^{n-r+1}(2^{2r}+1)-2] + j[2^{n-r+2}(2^{2r}+1)-2]\\ &\quad + k[2^{n-r+3}(2^{2r}+1)-2]\end{aligned}$$

$$= 2^{n-r}(2^{2r}+1)\tilde{A} - 2\tilde{B}$$

Using the relation, $M_{n+r} - M_{n-r} = 2^{n-r}(2^{2r}-1)$ we get

$$\begin{aligned}\widetilde{M_{n+r}} - \widetilde{M_{n-r}} &= (M_{n+r}-M_{n-r}) + i(M_{n+r+1}-M_{n-r+1}) + j(M_{n+r+2}-M_{n-r+2})\\ &\quad + k(M_{n+r+3}-M_{n-r+3})\\ &= [2^{n-r}(2^{2r}-1)] + i[2^{n-r+1}(2^{2r}-1)] + j[2^{n-r+2}(2^{2r}-1)] + k[2^{n-r+3}(2^{2r}-1)]\\ &= 2^{n-r}(2^{2r}-1)\tilde{A}\end{aligned}$$

Theorem 7. Let $n \geq 1, r \geq 1$ be integers. Then

i. $\widetilde{ML_{n+1}} + \widetilde{ML_n} = 3(2^n)\tilde{A} + 2\tilde{B}$

ii. $\widetilde{ML_{n+1}} - \widetilde{ML_n} = 2^n\tilde{A}$

iii. $\widetilde{ML_{n+r}} + \widetilde{ML_{n-r}} = 2^{n-r}(2^{2r}+1)\tilde{A} + 2\tilde{B}$

iv. $\widetilde{ML_{n+r}} - \widetilde{ML_{n-r}} = 2^{n-r}(2^{2r}-1)\tilde{A}$

where $\tilde{A} = 1 + 2i + 2^2 j + 2^3 k$, $\tilde{B} = 1 + i + j + k$

Proof. By using (1.2) and the identity $ML_{n+1} + ML_n = 3(2^n) + 2$, we have

$$\widetilde{ML_{n+1}} + \widetilde{ML_n} = (ML_{n+1} + ML_n) + i(ML_{n+2} + ML_{n+1}) + j(ML_{n+3} + ML_{n+2}) + k(ML_{n+4} + ML_{n+3})$$

$$= 3(2^n)[1 + 2i + 2^2 j + 2^3 k] + 2(1 + i + j + k)$$

$$= 3(2^n)\tilde{A} + 2\tilde{B}$$

By using the relation $ML_{n+1} - ML_n = 2^n$,

$$\widetilde{ML_{n+1}} - \widetilde{ML_n} = (ML_{n+1} - ML_n) + i(ML_{n+2} - ML_{n+1}) + j(ML_{n+3} - ML_{n+2}) + k(ML_{n+4} - ML_{n+3})$$

$$= 2^n[1 + 2i + 2^2 j + 2^3 k]$$

$$= 2^n\tilde{A}$$

By defining, $\widetilde{ML_{n+r}} = ML_{n+r} + iML_{n+r+1} + jML_{n+r+2} + kML_{n+r+3}$,

$\widetilde{ML_{n-r}} = ML_{n-r} + iML_{n-r+1} + jML_{n-r+2} + kML_{n-r+3}$ and using

$ML_{n+r} + ML_{n-r} = 2^{n-r}(2^{2r} + 1) + 2$, we obtain

$$\widetilde{ML_{n+r}} + \widetilde{ML_{n-r}} = (ML_{n+r} + ML_{n-r}) + i(ML_{n+r+1} + ML_{n-r+1}) + j(ML_{n+r+2} + ML_{n-r+2})$$
$$+k(ML_{n+r+3} + ML_{n-r+3})$$

$$= 2^{n-r}(2^{2r} + 1)[1 + 2i + 2^2 j + 2^3 k] + 2(1 + i + j + k)$$

$$= 2^{n-r}(2^{2r} + 1)\tilde{A} + 2\tilde{B}$$

Using the relation, $ML_{n+r} - ML_{n-r} = 2^{n-r}(2^{2r} - 1)$ we get

$$\widetilde{ML_{n+r}} - \widetilde{ML_{n-r}} = (ML_{n+r} - ML_{n-r}) + i(ML_{n+r+1} - ML_{n-r+1}) + j(ML_{n+r+2} - ML_{n-r+2})$$
$$+k(ML_{n+r+3} - ML_{n-r+3})$$

$$= 2^{n-r}(2^{2r} - 1)[1 + 2i + 2^2 j + 2^3 k]$$

$$= 2^{n-r}(2^{2r} - 1)\tilde{A}$$

Theorem 8. Let $n \geq 1$ be integer. Then

i. $\widetilde{M_n} + \widetilde{ML_n} = 2^{n+1}\tilde{A}$

ii. $\widetilde{ML_n} - \widetilde{M_n} = 2\tilde{B}$

iii. $3\widetilde{M_n} + \widetilde{ML_n} = 2\widetilde{M_{n+1}}$

where $\tilde{A} = 1 + 2i + 2^2 j + 2^3 k$, $\tilde{B} = 1 + i + j + k$

Proof. By using the identity, $\mathrm{M_n} + \mathrm{ML_n} = 2^{\mathrm{n+1}}$ we have

$$\widetilde{M_n} + \widetilde{ML_n} = (M_n + ML_n) + i(M_{n+1} + ML_{n+1}) + j(M_{n+2} + ML_{n+2}) + k(M_{n+3} + ML_{n+3})$$

$$= 2^{n+1}[1 + 2i + 2^2 j + 2^3 k] = 2^{n+1}\tilde{A}$$

By applying, $ML_n - M_n = 2$, we get

$$\widetilde{ML_n} - \widetilde{M_n} = (ML_n - M_n) + i(ML_{n+1} - M_{n+1}) + j(ML_{n+2} + M_{n+2}) + k(ML_{n+3} - M_{n+3})$$
$$= 2(1 + i + j + k) \quad = 2\tilde{B}$$

And substituting the identity $3M_n + ML_n = 2M_{n+1}$, we obtain

$$\widetilde{3M_n} + \widetilde{ML_n} = (3M_n + ML_n) + i(3M_{n+1} + ML_{n+1}) + j(3M_{n+2} + ML_{n+2}) + k(3M_{n+3} + ML_{n+3})$$
$$= 2(M_{n+1} + iM_{n+2} + jM_{n+3} + kM_{n+4}) \ = 2\widetilde{M_{n+1}}$$

Theorem 9. Let $n \geq 1$ be integer. Then

i. $\widetilde{M_n}\widetilde{ML_n} = 2 - 83(2^{2n}) + i(2^{2n+2} - 2^{n+3} - 2) + j(2^{2n+3} + 3(2^{n+2}) - 2) + k(2^{2n+4} - 2^{n+2} - 2)$

ii. $\widetilde{ML_n}\widetilde{M_n} = 2 - 83(2^{2n}) + i(2^{2n+2} + 2^{n+3} - 2) + j(2^{2n+3} - 3(2^{n+2}) - 2) + k(2^{2n+4} + 2^{2n+2} - 2)$

Proof. By multiplying (1.1) and (1.2) and using the identity $M_m ML_n + M_n ML_m = 2M_{m+n}$, $M_m ML_n - M_n ML_m = 2^{n+1} M_{m-n}$, we have

$$\widetilde{M_n}\widetilde{ML_n} = (M_n + iM_{n+1} + jM_{n+2} + kM_{n+3})(ML_n + iML_{n+1} + jML_{n+2} + kML_{n+3})$$

If necessary calculations are made, we obtain

$$\widetilde{\mathrm{M}_\mathrm{n}}\widetilde{\mathrm{ML}_\mathrm{n}} = 2 - 83(2^{2\mathrm{n}}) + \mathrm{i}(2^{2\mathrm{n}+2} - 2^{\mathrm{n}+3} - 2) + \mathrm{j}(2^{2\mathrm{n}+3} + 3(2^{\mathrm{n}+2}) - 2) + \mathrm{k}(2^{2\mathrm{n}+4} - 2^{\mathrm{n}+2} - 2)$$

In a similar way, we obtain

$$\widetilde{ML_n}\widetilde{M_n} = (ML_n + iML_{n+1} + jML_{n+2} + kML_{n+3})(M_n + iM_{n+1} + jM_{n+2} + kM_{n+3})$$
$$= 2 - 83(2^{2n}) + i(2^{2n+2} + 2^{n+3} - 2) + j(2^{2n+3} - 3(2^{n+2}) - 2) + k(2^{2n+4} + 2^{n+2} - 2)$$

The Mersenne and Mersenne-Lucas octonions:

Theorem 10. The generating functions for the Mersenne and Mersenne-Lucas octonions are

$$\check{f}(x) = \frac{\breve{M_0} + (\breve{M_1} - 3\breve{M_0})x}{1 - 3x + 2x^2}$$

$$\check{g}(x) = \frac{\breve{ML_0} + (\breve{ML_1} - 3\breve{ML_0})x}{1 - 3x + 2x^2}$$

Proof. Let us define $\check{f}(x) = \sum_{n=0}^{\infty} \breve{M_n} x^n$

Multiplying this equation by $1, 3x, 2x^2$ respectively and summing these equations, we obtain

$$(1 - 3x + 2x^2)\check{f}(x) = \breve{M_0} + (\breve{M_1} - 3\breve{M_0})x + (\breve{M_2} - \breve{M_2})x^2 + \cdots + (\breve{M_n} - \breve{M_n})x^n$$

$$\therefore \check{f}(x) = \frac{\breve{M_0} + (\breve{M_1} - 3\breve{M_0})x}{1 - 3x + 2x^2}$$

And define $\breve{g}(x) = \sum_{n=0}^{\infty} \breve{ML}_n x^n$

Multiplying this equation by $1, 3x, 2x^2$ respectively and summing these equations, we obtain

$(1 - 3x + 2x^2)\breve{g}(x) = \breve{ML}_0 + \left(\breve{ML}_1 - 3\breve{ML}_0\right)x + \left(\breve{ML}_2 - \breve{ML}_2\right)x^2 + \cdots + \left(\breve{ML}_n - \breve{ML}_n\right)x^n$

$$\therefore \breve{g}(x) = \frac{\breve{ML}_0 + \left(\breve{ML}_1 - 3\breve{ML}_0\right)x}{1 - 3x + 2x^2}$$

Theorem 11. The Binet formulas for the Mersenne and Mersenne-Lucas octonions are

$$\breve{M}_n = 2^n \breve{A} - \breve{B} \tag{3.1}$$

and $$\breve{ML}_n = 2^n \breve{A} + \breve{B} \tag{3.2}$$

where $\breve{A} = \sum_{s=0}^{7} 2^s e_s$, $\breve{B} = \sum_{s=0}^{7} e_s$.

Proof. From (1.3) and (1.4),

$\breve{M}_n = \sum_{s=0}^{7} M_{n+s} e_s = \sum_{s=0}^{7} (2^{n+s} - 1) e_s = 2^n \sum_{s=0}^{7} 2^s\, e_s - \sum_{s=0}^{7} e_s = 2^n \breve{A} - \breve{B}$

$\breve{ML}_n = \sum_{s=0}^{7} ML_{n+s} e_s = \sum_{s=0}^{7} (2^{n+s} + 1) e_s = 2^n \sum_{s=0}^{7} 2^s\, e_s + \sum_{s=0}^{7} e_s = 2^n \breve{A} + \breve{B}$

Lemma 2.

$\breve{A}\breve{B} = -253 + 47e_1 - 133e_2 - 73e_3 + 227e_4 + 87e_5 - 49e_6 + 155e_7$

and $\breve{B}\breve{A} = -253 - 41e_1 + 143e_2 + 91e_3 - 193e_4 - 21e_5 + 179e_6 + 103e_7$

Proof. From the definition of $\breve{A}$ and $\breve{B}$, and using the multiplication table for the basis of octonions, we computed these results.

Theorem 12 (Catalan Identity). Let $n, r \in \mathbb{Z}$, then we have

$\breve{M}_{n-r}\breve{M}_{n+r} - \breve{M}_n^{\ 2} = 2^{n-r}\left[(2^{r+1} - 1)\breve{A}\breve{B} - 2^{2r}\breve{B}\breve{A}\right]$

$\breve{ML}_{n-r}\breve{ML}_{n+r} - \breve{ML}_n^{\ 2} = 2^{n-r}\left[(1 - 2^{r+1})\breve{A}\breve{B} + 2^{2r}\breve{B}\breve{A}\right]$

where $\breve{A} = \sum_{s=0}^{7} 2^s e_s$, $\breve{B} = \sum_{s=0}^{7} e_s$.

Proof. By using (3.1) and (3.2), we obtain

$$\begin{aligned}
\breve{M}_{n-r}\breve{M}_{n+r} - \breve{M}_n^{\ 2} &= \left(2^{n-r}\breve{A} - \breve{B}\right)\left(2^{n+r}\breve{A} - \breve{B}\right) - \left(2^n\breve{A} - \breve{B}\right)^2 \\
&= 2^{2n}\breve{A}^2 - 2^{n+r}\breve{B}\breve{A} - 2^{n-r}\breve{A}\breve{B} + \breve{B}^2 - 2^{2n}\breve{A}^2 - \breve{B}^2 + 2^{n+1}\breve{A}\breve{B} \\
&= 2^{n-r}\left[(2^{r+1} - 1)\breve{A}\breve{B} - 2^{2r}\breve{B}\breve{A}\right]
\end{aligned}$$

$$\begin{aligned}
\breve{ML}_{n-r}\breve{ML}_{n+r} - \breve{ML}_n^{\ 2} &= \left(2^{n-r}\breve{A} + \breve{B}\right)\left(2^{n+r}\breve{A} + \breve{B}\right) - \left(2^n\breve{A} + \breve{B}\right)^2 \\
&= 2^{2n}\breve{A}^2 + 2^{n+r}\breve{B}\breve{A} + 2^{n-r}\breve{A}\breve{B} + \breve{B}^2 - 2^{2n}\breve{A}^2 - \breve{B}^2 + 2^{n+1}\breve{A}\breve{B} \\
&= 2^{n-r}\left[(1 - 2^{r+1})\breve{A}\breve{B} + 2^{2r}\breve{B}\breve{A}\right]
\end{aligned}$$

Since Cassini's identity is a special case of Catalan's identity, we get the below result by substituting $r = 1$ in Catalan's identity.

Theorem 13 (Cassini Identity). For any integer n,

$$\widetilde{M_{n-1}}\widetilde{M_{n+1}} - \widetilde{M_n}^2 = 2^{n-1}\left[3\breve{A}\breve{B} - 4\breve{B}\breve{A}\right]$$

$$\widetilde{ML_{n-1}}\widetilde{ML_{n+1}} - \widetilde{ML_n}^2 = 2^{n-1}\left[4\breve{B}\breve{A} - 3\breve{A}\breve{B}\right]$$

where $\breve{A} = \sum_{s=0}^{7} 2^s e_s$, $\breve{B} = \sum_{s=0}^{7} e_s$.

Theorem 14 (d'Ocagne's Identity). For any integers n, m, we have

$$\widetilde{M_m}\widetilde{M_{n+1}} - \widetilde{M_{m+1}}\widetilde{M_n} = 2^m\breve{A}\breve{B} - 2^n\breve{B}\breve{A}$$

$$\widetilde{ML_m}\widetilde{ML_{n+1}} - \widetilde{ML_{m+1}}\widetilde{ML_n} = 2^n\breve{B}\breve{A} - 2^m\breve{A}\breve{B}$$

where $\breve{A} = \sum_{s=0}^{7} 2^s e_s$, $\breve{B} = \sum_{s=0}^{7} e_s$.

Proof.

$$\widetilde{\mathrm{M}_\mathrm{m}}\widetilde{\mathrm{M}_{\mathrm{n+1}}} - \widetilde{\mathrm{M}_{\mathrm{m+1}}}\widetilde{\mathrm{M}_\mathrm{n}} = \left(2^\mathrm{m}\breve{\mathrm{A}} - \breve{\mathrm{B}}\right)\left(2^{\mathrm{n+1}}\breve{\mathrm{A}} - \breve{\mathrm{B}}\right) - \left(2^{\mathrm{m+1}}\breve{\mathrm{A}} - \breve{\mathrm{B}}\right)\left(2^\mathrm{n}\breve{\mathrm{A}} - \breve{\mathrm{B}}\right)$$

$$= 2^{m+n+1}\breve{A}^2 - 2^{n+1}\breve{B}\breve{A} - 2^m\breve{A}\breve{B} + \breve{B}^2 - 2^{m+n+1}\breve{A}^2 - \breve{B}^2 + 2^{m+1}\breve{A}\breve{B} + 2^n\breve{B}\breve{A}$$

$$= 2^m\breve{A}\breve{B} - 2^n\breve{B}\breve{A}$$

$$\widetilde{ML_m}\widetilde{ML_{n+1}} - \widetilde{ML_{m+1}}\widetilde{ML_n} = \left(2^m\breve{A} + \breve{B}\right)\left(2^{n+1}\breve{A} + \breve{B}\right) - \left(2^{m+1}\breve{A} + \breve{B}\right)\left(2^n\breve{A} + \breve{B}\right)$$

$$= 2^{m+n+1}\breve{A}^2 + 2^{n+1}\breve{B}\breve{A} + 2^m\breve{A}\breve{B} + \breve{B}^2 - 2^{m+n+1}\breve{A}^2 - \breve{B}^2 - 2^{m+1}\breve{A}\breve{B} - 2^n\breve{B}\breve{A}$$

$$= 2^n\breve{B}\breve{A} - 2^m\breve{A}\breve{B}$$

Theorem 15. For any integer n,

$$\widetilde{\mathrm{M}_\mathrm{n}} + \widetilde{\mathrm{ML}_\mathrm{n}} = 2^{\mathrm{n+1}}\breve{\mathrm{A}}$$

$$\widetilde{M_n} - \widetilde{ML_n} = -2\breve{B}$$

Proof. By using (1.3), (1.4) and the identities, $M_n + ML_n = 2^{n+1}$ and $M_n - ML_n = -2$

$$\begin{aligned}\widetilde{M_n} + \widetilde{ML_n} &= (M_n + ML_n)e_0 + (M_{n+1} + ML_{n+1})e_1 + (M_{n+2} + ML_{n+2})e_2 + (M_{n+3} + ML_{n+3})e_3 \\ &\quad + (M_{n+4} + ML_{n+4})e_4 + (M_{n+5} + ML_{n+5})e_5 + (M_{n+6} + ML_{n+6})e_6 + (M_{n+7} \\ &\quad + ML_{n+7})e_7 \\ &= 2^{n+1}(e_0 + 2e_1 + 2^2e_2 + 2^3e_3 + 2^4e_4 + 2^5e_5 + 2^6e_6 + 2^7e_7) \\ &= 2^{n+1}\breve{A}\end{aligned}$$

$$\begin{aligned}\widetilde{M_n} - \widetilde{ML_n} &= (M_n - ML_n)e_0 + (M_{n+1} - ML_{n+1})e_1 + (M_{n+2} - ML_{n+2})e_2 + (M_{n+3} - ML_{n+3})e_3 \\ &\quad + (M_{n+4} - ML_{n+4})e_4 + (M_{n+5} - ML_{n+5})e_5 + (M_{n+6} - ML_{n+6})e_6 + (M_{n+7} \\ &\quad - ML_{n+7})e_7 \\ &= -2(e_0 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \\ &= -2\breve{B}\end{aligned}$$

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