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Research Article

Zero Divisor Graph of a Commutative Ring

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Abstract

The main aim is to relate the theoretic properties of a commutative ring with properties of graph. For a commutative ring, the set of zero-divisors of denoted by Z(R). A simple graph Γ(R) is associated with the vertices which are non zero zero-divisors denoted by Z(R)*=Z(R)-{0}, where for distinct non zero zero-divisors of R x, y, the vertices x and y are connected by an edge if x y=0.This study illustrates the structure of Γ(R) and the properties of Z(R). We study when Γ(R) can be a complete graph and a star graph and examine the connectivity and diameter and grith of the graph Γ(R). We also study Γ(R) for non-isomorphic rings. The properties of Γ(R), for a commutative ring R and If Z(R) is an annihilator ideal, and for a local ring R with maximal ideal M are given.

Keywords: *Isomorphic rings, annihilator ideal, local ring, maximal ideal, diameter, grith.*

Introduction

Beck, I. (1988) in [4], developed the concept of Zero-divisor graph for a commutative ring where he explained the concept of colorings. The same concept of coloring of a ring R which is commutative was given by Anderson, D.D. and Naseer, M. (1993) in [1].

The Zero-divisor graphs for semigroups was studied by DeMeyer, F.R. and McKenie, T. and Schneider, K. (2002) in [6] and by DeMeyer, F.R. and DeMeyer, L. (2005) in [7]. Dolzan and Polona Oblak (2011) in [5] studied on Zero-divisor graph of rings and semi rings.

This study illuminate the structure of $Z(R)$. Define for every pair of zero divisors x and y, if xy = 0 or x = y then x∼y. The relation ∼ usually is not transitive, but always reflexive and symmetric. This study proves that ∼ is transitive iff $\Gamma(R)$ is complete.

In Section 2, properties of $\Gamma(R)$ along with examples are discussed. In Section 3, Theorems and examples are discussed to show that $\Gamma(R)$ is a complete graph if it is a complete bipartite graph and if $\Gamma(R)$ is of the form $k_{1,n}$, a complete bipartite graph, then $\Gamma(R)$ is a star graph. Properties of $\Gamma(R)$ when R contains annihilator ideal, for a finite local ring R with maximal ideal M, then M = $Z(R)$ are discussed . In Section 4, $\Gamma(R)$ is connected with diam ≤ 3 and contains a cycle if g ($\Gamma(R)$) ≤ 7 are shown with examples

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Commutative property will be satisfied by many rings. $Z(R)$ denote the zero-divisors of R. The annihilator of a subset S over a ring r denoted by annR is the ideal formed by the elements of the ring that gives zero when multiplied by an element of S. $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$ and $\mathbb{F}_q$ are, the ring of integers, integers modulo n, rational numbers, and finite field with q elements respectively.

The reference for graph theory concepts are from [9]. A simple graph is a graph structure which has no multiple edges and loops with the vertex set V(G). G is reffered as connected, if there exists a path from one vertex to other vertex which are distinct. The length of the shortest path between any two vertices *x* to *y* is called *distance* between x and y denoted by *d(x, y)* (if there exists no path between the vertices x and y then *d(x, y)*=∞). *diam*(G) = sup{*d(x, y)/x* and *y* are vertices of G} is called the *diameter* of G. The length of a shortest cycle in is called as *grith* of G denoted by gr(G)

G (gr(G)= ∞ if G has no cycles). If any two distinct vertices of G are connected by an edge (adjacent), then G is said to be *complete graph*. If the vertex set of a graph G can be partitioned in to two disjoint subsets A and B such that two distinct vertices of G are connected by an edge if and only if they are in different vertex sets A and B is called *complete bipartite graph*, which is denoted by $K_{m,n}$, where number of vertices are $|A| = m$ and $|B| = n$ .If one of vertex set has only one element, then G is called a *star graph*, denoted by $K_{1,n}$.

Properties of $\Gamma(R)$

For a commutative ring R, Z(R) be set of zero divisors of R. To the ring R with the vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero-divisors of R. We draw a simple graph $\Gamma(R)$ with the vertices x and y are adjacent in $\Gamma(R)$, if x y = 0 for every pair of $x, y \in Z(R)^*$. Thus if R does not contain any zero divisors (Integral domain), then $\Gamma(R)$ is an empty graph. Hence we assume that R cannot be n integral domain.

In this section, Examples of $\Gamma(R)$ and behaviour of $\Gamma(R)$for non isomorphic rings are given.

Example 2.1. Given are $\Gamma(R)$ for some commutative rings.

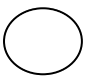

Figure 2.1. $\mathbb{Z}_4 or \mathbb{Z}_2[X]/[X^2]$

Figure 2.2. $\mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2 or\ \mathbb{Z}_3[X]/[X^2]$

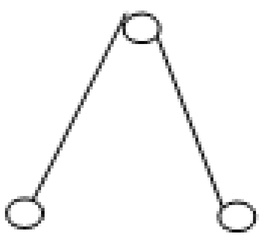

Figure 2.3. $\mathbb{Z}_8$

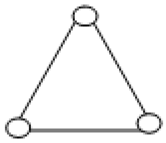

Figure 2.4. $\mathbb{Z}_2[X,Y]/[X^2, XY, Y^2\ ]$

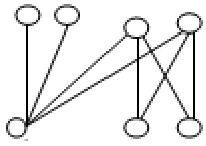

Figure 2.5. $\mathbb{Z}_{12}$

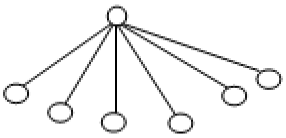

Figure 2.6. $\mathbb{Z}_{14}$

Non Isomorphic Rings may have same Structure of $\boldsymbol{\Gamma(R)}$

Structure of $\Gamma(R)$ for non isomorphic rings may be same. Thus the graph properties of two rings cannot decide the existence of isomorphism between them. This will be illustrated by an example.

$\mathbb{Z}_6$ and $\mathbb{Z}_8$ are non isomorphic rings but the zero-divisor graph of $\mathbb{Z}_6$ and $\mathbb{Z}_8$are given by

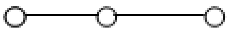

Figure 2.7. $\mathbb{Z}_8$={0,1,2,3,4,5,6,7}

Figure 2.8. $\mathbb{Z}_6 = \{0,1,,2,3,4,5\}$

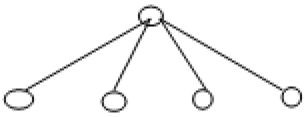

Figure 2.8. $\mathbb{Z}_{10}$={0,1,2,3,4,5,6,7,8,9}

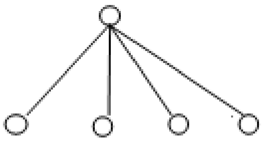

Figure 2.9. $\mathbb{Z}_2 \times \mathbb{Z}_5$

Graphs that may not be Realized as $\boldsymbol{\Gamma(R)}$ with Less than Four Vertices

All the graphs which are connected with atmost four vertices can be $\Gamma(R)$. Out of the six connected graphs having four vertices, the given three graphs can be $\Gamma(R)$.

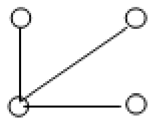

Figure 2.10. $\mathbb{Z}_2 \times \mathbb{F}_4$

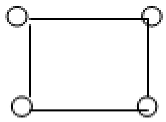

Figure 2.11. $\mathbb{Z}_3 \times \mathbb{Z}_3$

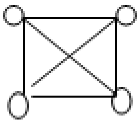

Figure 2.12. $\mathbb{Z}_{25}$ or $\frac{\mathbb{Z}_5[X]}{[X^2]}$

Proposition 2.2.1. Suppose Γ(R) with vertices {a, b, c, d} and edges a --- b, b----c, c----d cannot be realized as $\Gamma(R)$

Proof. Let Z(R) = {0, a, b, c, d} and are only the above zero divisor relations of a ring R. As $(a+c)b = 0$ therefore $a + c \in Z(R)$. Therefore $a + c$ is equal to one of 0, a, b, c or d. Simple check gives the $(a + c) = b$ as the only possibility. Similarly, $b + d = c$. Therefore $b\ is\ equal\ to\ a + c\ is\ equal\ to\ a + b + d$; so $a + d = 0$. Thus $bd = b(-a)$, a contradiction. For other two connected graphs having four vertices the proofs are similar.

For any n ≥ 5 $\Gamma(R)$ cannot be AN n-GON

$\Gamma(R)$ can always be a triangle or a square. For any $n \geq 5$ $\Gamma(R)$ cannot be a n-gon. But, for each $n \geq 3$, there exists a $\Gamma(R)$ with an n-cycle. Let $R_n = \mathbb{Z}_2[x_1, \ldots\ldots x_n] = \mathbb{Z}_2[X_1, \ldots\ldots X_n]/I$, where $I = (X_1^2, \ldots\ldots X_n^2, X_1X_2, X_2X_3, \ldots. X_nX_1)\ then\ \Gamma(R_n)$ is finite with a cycle of length n.

When $\Gamma(R)$ can be a Complete graph and a STAR Graph

Let R is a Cartesian product of two integral domains A and B denoted by A × B. Then $\Gamma(R)$ can be a complete bipartite graph where $|\Gamma(R)| = |A| + |B| - 2$.

Example 3.1.

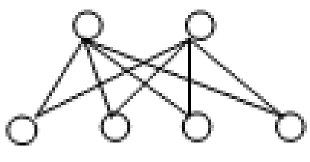

Figure 3.1. $\mathbb{Z}_3 \times \mathbb{Z}_5$
$|\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5)| = 3 + 5 - 2$

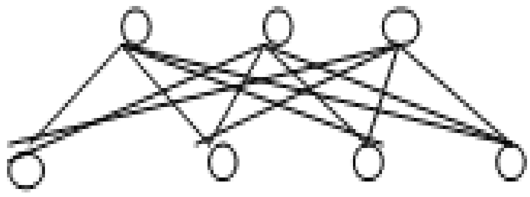

Figure 3.2. $\mathbb{Z}_4 \times \mathbb{Z}_5$
$|\Gamma(\mathbb{Z}_4 \times \mathbb{Z}_5)| = 4 + 5 - 2$

Theorem 3.1. For a commutative ring R. $\Gamma(R)$ is a complete graph if and only if either $R\ is\ isomorphic\ to\ the\ cartesian\ product\ of\ \mathbb{Z}_2 and\ \mathbb{Z}_2$ or product of x and y are equal to zero for all x, y ∈ $Z(R)$

Proof. Suppose $\Gamma(R)$ is complete, but there exist a $x \in Z(R)$ with $x^2 \neq 0$. To show that x^2 is equal to x. If not, then $x^3 = x^2x = 0$. Hence the product of x^2 and $(x + x^2)$ is equal to 0, with $x^2 \neq 0$, hence $x^2 \in Z(R)$. If $x + x^2 = x$ then $x^2 = 0$ which is a contradiction. This implies $x + x^2 \neq x$, so $x^2 = x^2x^3 = x(x + x^2) = 0$ but assumed that $\Gamma(R)$ is a complete, hence a contradiction. So $x^2 = x$. Let $xy = 0$ for every pair of $x, y \in Z(R)$. That implies the graph is complete. Hence $\Gamma(R)$ is complete.

When $\Gamma(R)$ can be a Star Graph

$\Gamma(R)$ is a star graph if $A = \mathbb{Z}_2$, with $|\Gamma(R)| = |B|$. For example $\Gamma(\mathbb{F}_p \times \mathbb{F}_q) = K_{p-1,q-1}$ and $\Gamma(\mathbb{F}_2 \times \mathbb{F}_q) = K_{1,q-1}$. Given are two examples.

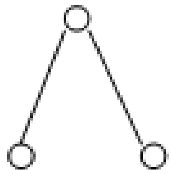

Figure 3.3. $\mathbb{Z}_2 \times \mathbb{Z}_3$

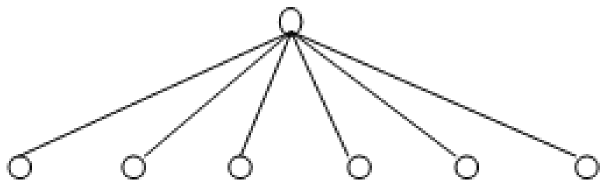

Figure 3.4. $\mathbb{Z}_2 \times \mathbb{Z}_7$

$\Gamma(R)$can be an infinite (i.e., a ring has no zero-divisors). If $\Gamma(R)$ is finite (i.e., a ring has finite number of zero-divisors), then $\Gamma(R)$ can be drawn. Often we restrict to the case such that R is a finite ring. Each element r of ring R is either a unit element or a zero-divisor only if R is finite, every prime ideal of R is an annihilator ideal, R is local if and only if all non unit of R is nilpotent. For some prime p and integer $n \geq 1$, char R = p^n if and only if R is a finite local ring with the maximal ideal M. Hence the maximal ideal which is equal to Z(R) is a p-group, therefore $|\Gamma(R)| = p^m - 1$ for a $m \geq 0$.

Theorem 3.2. *If either R is finite ring or an integral domain then $\Gamma(R)$is finite where R is a commutative ring.*

Proof. Let $\Gamma(R) = Z(R)^*$ is non empty and finite. This implies there exists a nonzero $x, y \in R$ with $xy = 0$. Suppose $I = ann(x)$, then $I \subseteq Z(R)$ is always finite and $ry \in I$ for all elements $r\ from\ R$. If R is infinite, then there exists an $i\ from\ I$ with $J = \{r \in R \setminus ry = i\}$ is infinite. Hence for $r, s \in J, (r-s)y = 0$, therefore $ann(y) \subseteq Z(R)$ is a infinite, and is a contradiction. Hence R is finite.

Theorem 3.3. If either $R \cong \mathbb{Z}_2 \times A$, where A is an integral domain, or Z(R)is an annihilator ideal then in $\Gamma(R)$there exists a vertex which is adjacent to every other vertex.

Proof. Let $Z(R)$ is not an annihilator ideal and let a be a nonzero element of $Z(R)$ which is having an edge to every other vertex. Now $a \notin ann(A) = I$, otherwise $Z(R) = I$ is an annihilator ideal. Hence I is the maximal among annihilator ideals and therefore it is a prime ideal. If$a^2 \neq a$, then $a^3 = a^2a = 0$, this implies $a \in I$, which is a contradiction. Thus $a^2 = a$: so $R = Ra \oplus R(1-a)$.

Therefore we suppose that $R = R_1 \times R_2$ with (1, 0) is having an edge to every other vertex. For any $1 \neq c \in R_1$, $(c, 0)$ is a zero divisor so$(c,0) = (c,0)(1,0) = (0,0)$ is a contradiction unless $c = 0$. Therefore $R_1 is\ somorphic\ to \mathbb{Z}_2$. If R_2 is not an integral domain, then there exists a non zero $b \in Z(R_2)$. Then (1, b) is a zero-divisor of R which is not adjacent to (1,0), a contradiction. Thus R_2 must be an integral domain, then there exists a non zero $b \in Z(R_2)$. This implies $(1, b)$ is zero-divisor of R which is not having an edge to (1, 0), a contradiction. Thus R_2 must be an integral domain. Among annihilator ideals, if $Z(R)$ is an annihilator ideal, then it is maximal and therefore is a prime. If $R \cong \mathbb{Z}_2 \times A$ which is an integral domain, then (1,0) will have an edge to every other vertex. If $Z(R) = ann(x)$ for a non zero $x \in R$, then x is connected by an edge with every other vertex.

Example 3.3

Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_7$. $\Gamma(R)$ is given below.

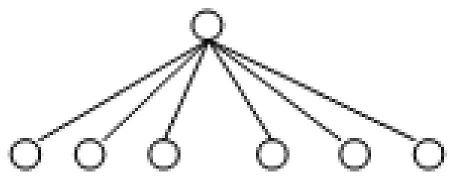

Figure 3.5. $\mathbb{Z}_2 \times \mathbb{Z}_7$

Example 3.4.

Let $R \cong \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. $Z(R) = \{0, 2, 4\} = ann(3)$ is an annihilator ideal hence 3 is connected by an edge to every other vertex of $\Gamma(R)$.

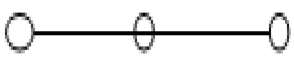

Figure 3.6. $\mathbb{Z}_6$

Both the cases above theorem will be for the same graph. R should be of the form $\mathbb{Z}_2 \times A$ for an integral domain if R is reduced and $\Gamma(R)$ has a vertex which is connected by an edge to all other vertices.

The proof of the Theorem 3.3 shows that if there is a vertex of $\Gamma(R)$ which is which is connected by an edge to every other vertex, then either x is idempotent with $Rx = \{0, x\}$ which is nothing but a prime ideal of R, or $Z(R) = ann(x)$ Let $Z(R)$ is an annihilator ideal, then $ann(Z(R)^*)$ is the set of vertices which are having edges to every other vertex.

Corollary 3.1. If R is local or if $R \cong \mathbb{Z}_2 \times F$, where F is finite field, then there is a vertex of $\Gamma(R)$ which is connected by an edge with every other vertex. Moreover, $|\Gamma| = |F| = p^n$ if $R \cong \mathbb{Z}_2 \times F$, and $|\Gamma| = p^n - 1$ if R is a local ring for some prime p and integer $n \geq 1$ for some commutative ring R.

Example 3.5.

Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_5$

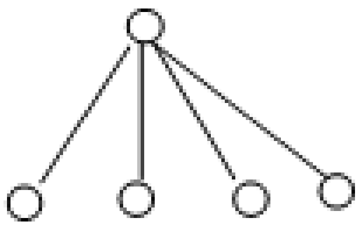

Figure 3.5. $\mathbb{Z}_2 \times \mathbb{Z}_5$

Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$

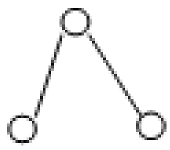

Figure 3.6. $\mathbb{Z}_2 \times \mathbb{Z}_3$

Example 3.6. Let $R \cong \mathbb{Z}_6$ is local, since M = {2, 3, 4} is the only maximal ideal of R $\Gamma(R)$ is given as

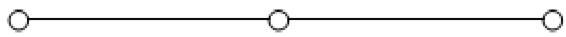

Figure 3.7. $\mathbb{Z}_6$

Diameter and Grith of $\boldsymbol{\Gamma(R)}$

All $\Gamma(R)$ are all connected and have small (≤ 3) diameter and grith. Hence, for not equal $x, y \in Z(R)^*$ either $xy = 0,\ xz = y = 0$ for a $z \in Z(R)^* - \{x, y\}, or\ xz_1 = z_1z_2 = z_2y = 0$ for some distinct $z_1, z_2 \in Z(R)^* - \{x, y\}$.

Theorem 4.1. Let R be a commutative ring. Always $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$, $gr(\Gamma(R)) \leq 7$, if $\Gamma(R)$ contains a cycle.

Proof. Suppose that $x, y \in Z(R)^*$ are distinct. If $xy = 0$, then $d(x, y) = 1$. Let xy is nonzero. If $x^2 = y^2 = 0$ then $x - xy - y$ length of the path is 2; hence $d(x, y) = 2$. If $x^2 = 0$ and $y^2 \neq 0$, then there exists $a, b \in Z(R)^* - \{x, y\}$ with $by = 0$. If $bx = 0$, then $x - b - y$ length of the path is 2. If $bx \neq 0$, then $x - bx - y$ length of the path is 2. In either case, $d(x, y) = 2$.

Hence a similar argument holds for $y^2 = 0$ and $x^2 \neq 0$. Thus we assume that x^2, xy, y^2 are all nonzero. Therefore $ax = by = 0$ for some $a, b \in Z(R)^* - \{x, y\}$ with. If $a = b$, then $x - a - y$ length of the path is 2. So that we may assume that $a \neq b$. If ab $= 0$ then $x - a - b - y$ length of the path is 3, thus $d(x, y) \leq 3$. If $ab \neq 0$ then $x - ab - y$ length of the path is 2, thus $d(x, y) = 2$. Therefore $d(x, y) \leq 3$ with $diam(\Gamma(R)) \leq 3$.

Example 4.1.

In $\mathbb{Z}_2 \times \mathbb{Z}_4$, the path. $(0, 1) - (1, 0) - (0, 2), (1, 0)$, shows that $diam(\Gamma(R)) = 3$.

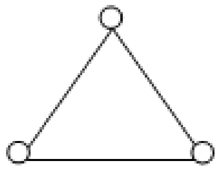

Figure 4.1. $\mathbb{Z}_2 \times \mathbb{Z}_4$

If $R \cong F \times K$ for finite fields F and K $|F|, |K| \geq 3$ for a finite commutative ring with then $gr(\Gamma(R)) = 4$.

Example 4.2.

Let $R \cong \mathbb{Z}_4 \times \mathbb{Z}_5$, where $\mathbb{Z}_5$ is a finite field with $|\mathbb{Z}_5| \geq 3$.

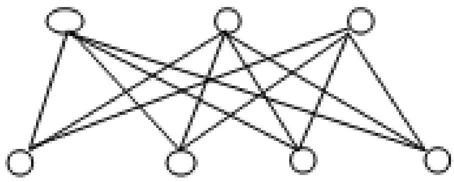

Figure 4.2. $\mathbb{Z}_4 \times \mathbb{Z}_5$

By the zero divisor graph $\Gamma(R)$, $gr(\Gamma(R)) = 4$.

Example 4.3.

Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ where $\mathbb{Z}_3$ and $\mathbb{Z}_5$ are finite fields with $|\mathbb{Z}_3|, |\mathbb{Z}_5| \geq 3$.

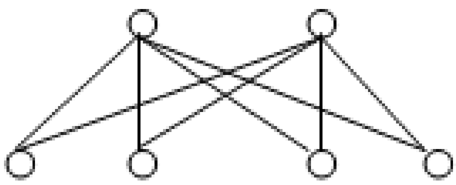

Figure 4.3. Z_3×Z_5

If either $|\Gamma(R)| \leq 2, |\Gamma(R)| = 3$ then, we can show that $gr(\Gamma(R)) = \infty$ and $\Gamma(R)$ is not complete.

Example 4.4.

Let $R \cong \mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6,7, 8\}$ and $\Gamma(R)$ is given below.

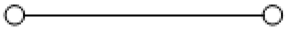

Figure 4.4. $\mathbb{Z}_9$

$|\Gamma(R)| = 2$, hence $gr\big(\Gamma(R)\big) = \infty$.

Example 4.5.

Let $R \cong \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $\Gamma(R)$ is given below.

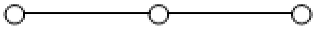

Figure 4.5. $\mathbb{Z}_6$

$|\Gamma(R)| = 3$, but not complete, hence $gr\big(\Gamma(R)\big) = \infty$.

Suppose that $R \cong \mathbb{Z}_2 \times A$ with $|Z(R)| = 2$ then $gr\big(\Gamma(R)\big) = \infty$. For each integer $n \geq 1$, let Γ_n be the graph with vertex set $\{x_1, \dots . x_n\}$ and $x_1 - x_2, x_2 - x_3, \dots . x_{n-1} - x_n$ as its only edges. By theorem 4.1, the "line graph" $\Gamma_n$can be realized as $|\Gamma(R_n)|$ if and only if $n \leq 3$.

Corollary 4.6. If R is a commutative ring then for $x, y \in Z(R)$, define $x{\sim}y$ if xy = 0 or x = y, and define $x \sim * \, y \; if \; xy = 0$.

(a) *if* $\Gamma(R)$ *is complete then relation* $\sim$ *is transitive which is an equivalence realtion.*
(b) *The relation* $\sim *$ *istransitive if and only if* $\Gamma(R)$ $is\ complete\ and\ R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$

Proof. Both the parts directly follow from Theorem 2.1. and Theorem 4.1.

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