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Research Article

# Common Fixed Point Theorem For Four Compatible And Subsequentially Continuous Maps In G-Metric Spaces

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**Abstract:**
In this manuscript, our aim is to prove a new common fixed point theorem for four compatible and subsequentially continuous (alternately sub compatible and reciprocally continuous) maps in the G-metric spaces satisfying a more generalized contractive condition.

**Keywords:** *Common fixed point, compatibility, G-metric spaces.*

## 1. Introduction:

The notion of G-metric spaces was introduced by Mustafa and Sims [6]. After that a lot of authors have worked in this direction [see 7-10].
Following definitions will be used in sequel:

**Definition 1.1[6] G-metric spaces:**

In 2006, Mustafa and Sims introduced the concept of G-metric space as follows:

Let $X$ be a nonempty set, and $let\ G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following:

(G1) $G(x,y,z) = 0\ if\ x = y = z,$
(G2) $0 < G(x,x,y)\ for\ all\ x,y\ in\ X\ with\ x \neq y,$
(G3) $G(x,x,y) \leq G(x,y,z)\ for\ all\ x,y,z\ in\ X\ with\ z \neq y,$
$(G4)$ $(x,z,y) = (\ G(x,y,z) = G(y,z,x) = \ldots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)\ for\ all\ x,y,z,a\ in\ X$ (rectangle inequality).

Then the function $G$ is called a $G$-metric on $X$ and the pair $(X,G)$ is called a $G$-metric space.

**Definition 1.2[6]** If $G(x,y,y) = G(y,x,x)\ \forall\ x,y \in X$, then $(X,G)$ is called a symmetric $G-$ metric space.

**Definition 1.3[1]** Let $(X,G)$ be a $G$- metric space and $S$ and $T$ be two self maps on $X$. Then $S$ and $T$ are said to be compatible if

$\lim_{n\rightarrow\infty} G\left(STx_{n,},TSx_n,TSx_n\right) = 0$ , whenever $\{x_n\}$ is a sequence in $X$ such that
$\lim_{n\rightarrow\infty} Sx_n = \lim_{n\rightarrow\infty} Tx_n = z$ for some $z \in X$.

**Definition 1.4[2]** Two self mappings $S$ and $T$ are said to be conditionally reciprocally continuous, if whenever the set of sequences $\{x_n\}$ in $X$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n$ is nonempty, there exit a sequence $\{y_n\}$ in $X$ satisfying $\lim_{n\to\infty} S\, y_n = \lim_{n\to\infty} Ty_n = t$ (say) such that $\lim_{n\to\infty} STy_n = St\ and\ \lim_{n\to\infty} TSy_n = Tt$.

**Definition 1.5[2]** A pair of self mappings $S$ and $T$ is said to be reciprocally continuous if $\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n = Tt$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ , for some $t\ in\ X$.

**Definition 1.6[5]** A pair of self mappings $S$ and $T$ is said to be subcompatible if $\lim_{n\to\infty} Sx_n = t$, $\lim_{n\to\infty} Tx_n = t$ whenever $\{x_n\}$ is a sequece in $X$ and $\lim_{n\to\infty} G\big(STx_{n,}, TSx_n, TSx_n\big) = 0$.

**Definition 1.7[5]** A pair of self mappings $S$ and $T$ is said to be subsequentially continuous if $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t\ in\ X$ such that

$\lim_{n\to\infty} STx_n = St$, $\lim_{n\to\infty} TSx_n = Tt$ .

**2. Main Result**

In this section, we shall prove a common fixed point Theorem for four compatible and subsequentially continuous self maps in G-metric spaces.

**Theorem 2.1.** Let $A, B, S\ and\ T$ be four self mappings on a G-metric space$(X, G)$, and suppose that the pairs $(A, S)$ and $(B, T)$ are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) and satisfying the following inequality:

$$G(Ax, By, Bz) \le p\{G(Sx, Ty, Tz) + G(Ax, Sx, Sx)\} + q\,\{G(Sx, Ty, Tz) + G(By, Ty, Tz)\}$$
$$+ \mathrm{r}\ max\left\{G(Sx, Ty, Tz),\ \frac{G(Sx,By,Bz)+ G\big((Ax,Ty,Tz)\big)}{2}\right\}, \qquad (2.1)$$

where $p, q, r > 0\ and\ p + q + r < 1$.

Then $A, B, S\ and\ T$ have a unique common fixed point in $X$.

**Proof.** Given that the pair $(A, S)$ is sequentially continuous and compatible, so there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and $\lim_{n\to\infty} G\big(ASx_{n,}, SAx_n, SAx_n\big) = G(\mathrm{Az}, \mathrm{Sz}, \mathrm{Sz}) = 0$.

This implies that $Az = Sz$.

Thus $z$ is a coincidence point of the pair $(\ A, S)$.

Similarly, the pair $(B, T)$ is sequentially continuous and compatible, so there exists a sequence $\{\mathrm{y_n}\}$ in $X$ such that

$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w$ for some $w \in X$ and

$\lim_{n\to\infty} G\big(BTy_{n,}, TBy_n, TBy_n\big) = G(\mathrm{Bw}, \mathrm{Tw}, \mathrm{Tw}) = 0$.

This implies that $Bw = Tw$, that is, $w$ is a coincidence point of the pair $(B, T)$.

Now, we claim that $z = w,\ if\ z \ne w$, then using the inequality (2.1) with $x = x_n, y = y_{n.}\ and\ z = y_n$, We have

$$G(A\mathrm{x_n}, By_n, By_n) \le p\{G(Sx_n, Ty_n, Ty_n) + G(Ax_n, Sx_n, Sx_n)\}$$
$$+q\,\{G(Sx_n, Ty_n, Ty_n) + G(By_n, Ty_n, Ty_n)\}$$
$$+\,r\ max\left\{G(Sx_n, Ty_n, Ty_n),\ \frac{G(Sx_n,By_n,By_n)+ G\big((A\mathrm{x_n},Ty_n,Ty_n)\big)}{2}\right\}.$$

Making $n \to \infty$, we get

$$G(z, w, w) \le p\{G(z, w, w) + G(z, z, z)\} + q\,\{G(z, w, w) + G(z, z, z)\}$$

$$+ r\, max\left\{G(z,w,w),\ \frac{G(z,w,w)+G\big((z,w,w)\big)}{2}\right\}, \text{ that is,}$$

$$G(z,w,w) \le p\{G(z,w,w) + 0\} + q\{G(z,w,w) + 0\}$$
$$+ r\, max\{G(z,w,w),\ G(z,w,w)\}, \text{ that is,}$$

$$G(z,w,w) \le p\{G(z,w,w)\} + q\{G(z,w,w)\}$$
$$+ r\{G(z,w,w)\}$$
$$< (p+q+r)\{G(z,w,w)\}$$
$$< \{G(z,w,w)\},$$

a contradiction.

Hence $z = w$.

Now, we prove that $Az = z$.

On the contrary, suppose that, $Az \neq z$.

On making use of the inequality (2.1) with $x = z, y = y_n, z = y_n$ , we have

$$G(Az, By_n, By_n) \le p\{G(Sz, Ty_n, Ty_n) + G(Az, Sz, Sz)\}$$
$$+q\{G(Sz, Ty_n, Ty_n) + G(By_n, Ty_n, Ty_n)\}$$
$$+ r\, max\left\{G(Sz, Ty_n, Ty_n),\ \frac{G(Sz,By_n,By_n)+G\big((Az,Ty_n,Ty_n)\big)}{2}\right\}.$$

Making limit as $n \to \infty$, we get

$$G(Az,w,w) \le p\{G(Sz,w,w) + G(Az,Az,Az)\} + q\{G(Sz,w,w) + G(w,w,w)\}$$
$$+ r\, max\left\{G(Sz,w,w),\ \frac{G(Sz,w,w)+G\big((Sz,w,w)\big)}{2}\right\}, \text{ that is,}$$

$$G(Az,w,w) \le p\{G(Sz,w,w) + 0\} + q\{G(Sz,w,w) + 0\}$$
$$+ r\, max\{G(Sz,w,w),\ G(Sz,w,w)\}, \text{ that is,}$$

$$G(Az,w,w) \le p\{G(Sz,w,w)\} + q\{G(Sz,w,w)\}$$
$$+ r\{G(Sz,w,w)\}$$
$$< (p+q+r)\{G(Sz,w,w)\}$$
$$< \{G(Az,w,w)\},$$

a contradiction.

Hence $Az = w = z$.

So $Az = z$.

Now, we claim that $Bz = z$.

Let, if possible, $Bz \neq z$.

Using the inequality (2.1) with $x = x_n,\ y = z$.

$$G(Ax_n, Bz, Bz) \le p\{G(Sx_n, Tz, Tz) + G(Ax_n, Sx_n, Sx_n)\}$$
$$+q\{G(Sx_n, Tz, Tz) + G(Bz, Tz, Tz)\}$$
$$+ r\, max\left\{G(Sx_n, Tz, Tz),\ \frac{G(Sx_n,Bz,Bz)+G\big((Ax_n,Tz,Tz)\big)}{2}\right\}.$$

Making limit as $n \to \infty$, we get

$$G(z,Bz,Bz) \le p\{G(Sz,Tz,Tz) + G(z,z,z)\} + q\{G(z,Tz,Tz) + G(Bz,Tz,Tz)\}$$
$$r\, max\left\{G(z,Tz,Tz),\ \frac{G(z,Bz,Bz)+G\big((z,Tz,Tz)\big)}{2}\right\}.$$

$$G(z,Bz,Bz) \le p\{G(z,Bz,Bz) + 0\} + q\{G(z,Bz,Bz) + 0\}$$
$$+ r\, max\left\{G(z,Bz,Bz),\ \frac{G(z,Bz,Bz)+G\big((z,B,Bz)\big)}{2}\right\}.$$

$$G(z,Bz,Bz) \le p\{G(z,Bz,Bz)\} + q\{G(z,Bz,Bz)\} + r\{G(z,Bz,Bz)\}$$

$$G(z,Bz,Bz) \le (p+q+r)\{G(z,Bz,Bz)\}$$

$G(z, Bz, Bz) < G(z, Bz, Bz)$,
a contradiction.
Hence $Bz = z$.
So $Az = Bz = Sz = Tz = z$.
Hence $z$ is a common fixed point of the four mappings $A, B, S \; and \; T$.

**Uniqueness:** Let $w$ be another common fixed point of the mappings $A, B, S \; and \; T$. Then we have $Aw = Bw = Sw = Tw = w$. Now using the inequality (2.1) we have.

$$G(z,w,w) = G(\mathrm{Az},\mathrm{Bw},\mathrm{Bw}) \le p\{G(Sz,Tw,Tw) + G(Az,Sz,Sz)\} + q\{G(Sz,Tw,Tw) + G(Bw,Tw,Tw)\} + r\, max\left\{G(Sz,Tw,Tw), \frac{G(Sz,Bw,Bw)+G\big((Az,Tw,Tw)\big)}{2}\right\}.$$

$$G(z,w,w) \le p\{G(z,w,w) + G(z,z,z)\} + q\{G(z,w,w) + G(w,w,w)\} + r\, max\left\{G(z,w,w), \frac{G(z,w,w)+G\big((z,w,w)\big)}{2}\right\}.$$

$G(z,w,w) \le p\{G(z,w,w)\} + q\{G(z,w,w)\} + r\, max\{G(z,w,w),\}$, that is,
$G(z,w.w) \le (p+q+r)\big(G(z,w,w)\big)$, that is,
$G(z,w.w) < \big(G(z,w,w)\big)$,
a contradiction. Hence $z$ is a unique common fixed point of the four mappings $A, B, S \; and \; T$.

Now as our supposition the pair $(A, S)$ is subcompatible and reciprocally continuous, then there exists a sequence $\{x_n\}$ in $X$ such that
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$ and
$\lim_{n\to\infty} G\big(ASx_n, SAx_n, SAx_n\big) = G(\mathrm{Az},\mathrm{Sz},\mathrm{Sz}) = 0$. This implies that $Az = Sz$. That is $z$ is a coincidence point of the pair$(A, S)$

Similarly, as our supposition that the pair $(B, T)$ is reciprocally continuous and subcompatible, then there exists a sequence $\{\mathrm{y_n}\}$ in $X$ such that
$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w$ for some $z \in X$ and $\lim_{n\to\infty} G\big(BTy_n, TBy_n, TBy_n\big) = G(\mathrm{Bw},\mathrm{Tw},\mathrm{Tw}) = 0$. This implies that $Bw = Tw$. That is $w$ is a coincidence point of the pair$(B, T)$. The remaining proof is as follows from the upper part.

## References


1. B. S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, Math.Comput. Modelling (2011) doi:10.1016/j.mcm2011.01.036.
2. D. K. Patel, P. Kumam, & D. Gopal, Some discussion on the existence of common fixed points for a pair of maps. Fixed Points Theorey Appl 2013, **187**(2013).
3. G. Jungck , Compatible mappings and common fixed points for non continuous non self maps on non metric spaces, Far East J. Math.Sci. **4**(2)1996, 199-215.
4. G. Jungck, Compatible mappings and common fixed points, Int. J.Math. Sci. **9**(4)1986 771-779
5. H. Bouhadjera and C. Godet-Thobie, Common fixed point theorems for pairs of subcompatible maps. Old version, 2009, http://arxiv.org/abs/0906.3159.
6. Z. Mustafa and B. Sims, A new approach to generalized metric spaces, Journal of Nonlinear Convex Analysis, 7 (2006), 289-297.
7. Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces, Fixed Point Theory and Applications, Volume 2009, Article ID 917175.

**8.** Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, Fixed Point Theory and Applications, **2008**(2008), Article ID 189870.

**9.** Z. Mustafa, W. Shatanawi and M. Bataineh, Fixed point theorem on uncomplete G-metric spaces, Journal of Mathematics and Statistics, **4**(2008), 196-201.

**10.** Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces, Int. J. of Math. and Math. Sci., **2009**(2009), Article ID 283028.